Quantum Field Theory II
Exercise sheet 6

Exercise 6.1 Scalar QCD

Consider scalar QCD, a theory of complex scalar fields interacting with $SU(3)$ gauge bosons. Interactions like these are present in supersymmetric extensions of the Standard Model.
The Lagrangian is given by

$$\mathcal{L} = (D_\mu\phi)^\dagger (D^\mu\phi) - m^2\phi^\dagger\phi - \frac{1}{4}G^{\mu\nu,a}G_{\mu\nu,a} \ ,$$

where ϕ is in a representation R of $SU(3)$ with generators T_R^a and

$$\begin{aligned} G^{\mu\nu,a} &= \partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} + gf^{abc}A^{\mu,b}A^{\nu,c} \ , \\ D^\mu &= \partial^\mu - igT_R^a A^{\mu,a} \ . \end{aligned}$$

The goal of this exercise is to derive the Feynman rules in the gauge $\partial^\mu A_\mu^a = 0$.

a) Add the gauge-fixing term for R_ξ gauge, $\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}\partial^\mu A_\mu^a \partial^\nu A_\nu^a$, the ghost Lagrangian, $\mathcal{L}_{\text{ghost}} = -\bar{c}^a\partial^\mu D_\mu^{ab}c^b$ and source terms for $A^{\mu,a}$ and for ϕ, $\phi^\dagger$ to $\mathcal{L}$. Then you can write

$$Z[J_A, J_\phi, J_{\phi^\dagger}] \propto \int \mathcal{D}A\,\mathcal{D}\phi\,\mathcal{D}\phi^\dagger\,\mathcal{D}\bar{c}\,\mathcal{D}c\ \exp\left(iS_{\text{free}} + iS_{\text{int}} + iS_{\text{src}}\right)$$

where S_{free} only contains non-interacting terms quadratic in the fields and terms with external sources are collected in S_{src}. State S_{free}, S_{int} and S_{src} explicitly.
Remember: $D_\mu^{ab} = \delta^{ab}\partial_\mu - igA_\mu^c(T_A^c)^{ab}$, T_A being in the adjoint representation of $SU(3)$.

b) Rewrite S_{free} and S_{int} in momentum space.

c) Determine the propagators from S_{free} using known results from the lecture.

d) What are the interaction vertices?

Exercise 6.2 BRST transformation

As seen in the previous exercise, the path integral for a non-abelian gauge theory is quadratic in the gauge-fixing term:

$$Z \sim \int \mathcal{D}A_\mu^a \mathcal{D}\overline{\psi}^i \mathcal{D}\psi^i \mathcal{D}\overline{\eta}^a \mathcal{D}\eta^a\ e^{i\int d^4x\left[\mathcal{L}_{YM} + \mathcal{L}_{fermions} + \mathcal{L}_{ghost} - \frac{1}{2\xi}(\mathcal{G}^a(A_\mu^a))^2\right]}$$

It can be made linear in the gauge-fixing term by introducing a new bosonic field w^a, and one recovers the previous path integral by integrating out the field w^a.
In this formalism, the exponent in the path integral is not gauge-invariant, but is invariant under the so-called BRST symmetry. Under this symmetry, the fields transform proportionally to an infinitesimal parameter θ as

$$\begin{aligned} \delta_\theta A_\mu^a &= -\frac{\theta}{g}D_\mu^{ab}\eta^b \\ \delta_\theta \eta^a &= \frac{\theta}{2}f^{abc}\eta^b\eta^c \end{aligned}$$

Prove that two successive BRST transformations leave the gauge field invariant:

$$\delta_{\theta_1}\delta_{\theta_2}A_\mu^a = 0$$