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Quantum Field Theory II
Exercise sheet 5

FS 10
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Exercise 5.1 Superexercise!

Let us introduce the notion of superspace. Coordinates in superspace are commuting and anti-commuting numbers, $z = (z_1, \ldots, z_{N_B})^T$ and $\eta = (\eta_1, \ldots, \eta_{N_F})^T$ respectively (in supersymmetry the ordinary space dimensions correspond to bosonic degrees of freedom, the anticommuting ones to fermionic degrees of freedom). We can define a linear transformation in superspace

$$\begin{pmatrix} z' \\ \eta' \end{pmatrix} = \begin{pmatrix} A & D \\ C & B \end{pmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix} = M \begin{pmatrix} z \\ \eta \end{pmatrix} \quad ,$$

where A and B are commuting matrices (i.e. they have commuting entries), C and D anticommuting (Grassmann entries).

a) Let us define the superdeterminant of a superspace matrix M as

$$(\det M)^{-1} = \int \prod_{\substack{i=1,\ldots,N_B \\ j=1,\ldots,N_F}} \frac{\mathrm{d}z_i^* \mathrm{d}z_i}{2\pi} \mathrm{d}\eta_j^* \mathrm{d}\eta_j \exp[-z^\dagger A z - z^\dagger D \eta - \eta^\dagger C z - \eta^\dagger B \eta] \quad .$$

Starting from this definition show that

$$\det M = \frac{\det A}{\det (B - C A^{-1} D)} \quad .$$

Hint: perform the shift of the integration variables

$$\begin{aligned} z &= z' - A^{-1} D \eta \\ z^\dagger &= z^{\dagger\prime} - \eta^\dagger C A^{-1} \quad . \end{aligned}$$

Once you have done so, you'll be able to separate the integral over the ordinary c-numbers and the integral over the Grassmann variables. Remember that Gaussian integrals in the two cases are different!

b) Show that the supertrace defined as

$$\mathrm{Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, B$$

satisfies the cyclicity property

$$\mathrm{Tr}[M_1 M_2] = \mathrm{Tr}[M_2 M_1] \quad .$$

c) Show that

$$\mathrm{Tr}\ln M_1 M_2 = \mathrm{Tr}\ln M_1 + \mathrm{Tr}\ln M_2 \quad .$$

Hint: use the Campbell-Baker-Hausdorff formula

$$\exp(A)\exp(B) = \exp\left\{ A + B + \frac{1}{2!}[A,B] + \frac{1}{3!}\left(\frac{1}{2}[[A,B],B] + \frac{1}{2}[A,[A,B]]\right) + \ldots \right\}$$

and the cyclicity property you proved at point b).

d) Writing

$$M = \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & A^{-1}D \\ 0 & B - C A^{-1} D \end{pmatrix}$$

and using property c), prove that

$$\ln \det M = \mathrm{Tr}\ln M$$

and

$$\det(M_1 M_2) = \det(M_1)\det(M_2) \quad .$$

e*) Write the Jacobian of a transformation in superspace.

Exercise 5.2   Path integral in gauge theories

Consider the Yang-Mills theory

$$\mathcal{L}_{YM} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a}$$

where $G^a_{\mu\nu}$ is the field strength tensor defined as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

This theory is invariant under gauge transformations

$$A_\mu \rightarrow A'_\mu = U(x)A_\mu U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

with $A_\mu \equiv A^a_\mu T^a$ and the T^a are the generators of the gauge group. $U(x)$ can be parametrised as

$$U(x) = e^{ig\theta^a(x)T^a}$$

The generating functional for this theory is

$$Z[J^{\mu,a}] = Z[0]\int \mathcal{D}A^a_\mu \exp\left[i\int d^4x \left(\mathcal{L}_{YM} + J^{\mu,a}A^a_\mu\right)\right]$$

Prove that the integration measure is gauge invariant, namely $\mathcal{D}A^a_\mu = \mathcal{D}A'^a_\mu$.

Hint: The measure transforms as:

$$\mathcal{D}A'^a_\mu = \mathcal{D}A^a_\mu \det\left(\frac{\delta A'^a_\mu}{\delta A^b_\mu}\right).$$