

**Exercise 3.1 Basics of QFT**

Consider the Lagrangian for a real scalar field  $\phi(x)$ :

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

- a) The principle of least action states that the action  $S = \int d^4x \mathcal{L}$  has to be an extremum, so that small variations vanish ( $\delta S = 0$ ). Use this fact to derive the Euler-Lagrange equation

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Compute it explicitly for the real scalar field.

- b) Use canonical quantisation to derive the Feynman propagator  $D_F(x - y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle$ .  
*Hint: first compute the Hamiltonian of the system, apply the usual commutation relations, and then rewrite the field  $\phi$  in terms of creation and annihilation operators.*

**Exercise 3.2 Gaussian integrals**

Using the known result for the Gaussian integral

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2} a x^2} = \sqrt{\frac{2\pi}{a}}$$

show that

$$\int_{-\infty}^{\infty} dx_1 \dots dx_N e^{-\frac{1}{2} x^T A x} = \sqrt{\frac{(2\pi)^N}{\det A}}$$

where  $x^T = (x_1, \dots, x_N)$  and  $A$  is a symmetric, positive definite  $N \times N$  matrix.

*Remark: taking the limit  $N \rightarrow \infty$ , one can see that this identity holds formally for operators, so that for the action*

$$S_0 = -\frac{1}{2} \int d^4x \phi (\partial^2 + m^2) \phi$$

one has

$$\int \mathcal{D}\phi e^{iS_0} = \text{const} \cdot [\det(\partial^2 + m^2)]^{-1/2}$$

**Exercise 3.3 Discretisation of the path integral**

Consider the action for a free real scalar field of mass  $m$

$$S_0 = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right]$$

We want to evaluate exactly the path integral

$$\int \mathcal{D}\phi e^{iS_0}$$

in a finite volume  $V$ .

First, we replace the field  $\phi(x)$  defined on a continuum of points by variables  $\phi(x_i)$  defined at the points  $x_i$  of a square lattice. The integration measure becomes

$$\mathcal{D}\phi = \prod_i d\phi(x_i)$$

up to an irrelevant constant.

a) The field values  $\phi(x_i)$  can be represented by a discrete Fourier series:

$$\phi(x_i) = \frac{1}{V} \sum_n e^{-ik_n \cdot x_i} \phi(k_n)$$

where  $k_n^\mu = 2\pi n^\mu / L$  with  $n^\mu$  an integer, and  $V = L^4$ . Since  $\phi(x)$  is real the Fourier coefficients have to obey  $\phi^*(k) = \phi(-k)$ .

Rewrite the action  $S_0$  in terms of the Fourier coefficients  $\phi(k_n)$ .

b) Since the  $\phi(k_n)$  are complex, one can integrate separately their real and imaginary part. Show that the integration measure can be written

$$\mathcal{D}\phi(x) = \prod_{k_n^0 > 0} d\text{Re } \phi(k_n) d\text{Im } \phi(k_n)$$

and write the action  $S_0$  also in terms of  $\text{Re } \phi(k_n)$  and  $\text{Im } \phi(k_n)$ .

c) Using the results of exercise 2, perform the path integral. You should obtain

$$\int \mathcal{D}\phi e^{iS_0} = \prod_{k_n} \sqrt{\frac{-i\pi V}{m^2 - k_n^2}}$$

### Exercise 3.4 Four-point function

Consider the generating functional of the free Klein-Gordon theory

$$Z[J] = Z[0] \exp \left[ -\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) \right]$$

The four-point function is then

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \frac{1}{Z[0]i^4} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_3)} \frac{\delta}{\delta J(x_4)} Z[J] \Big|_{J=0}$$

Compute it and interpret the result pictorially.

*Hint: use the basic axiom of functional derivation in 4 dimensions*

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y)$$

For more complicated functions of  $J$ , one simply uses the ordinary Leibniz rule for derivatives of composite function, e.g

$$\frac{\delta}{\delta J(x)} J(y)J(z) = \left( \frac{\delta}{\delta J(x)} J(y) \right) J(z) + J(y) \left( \frac{\delta}{\delta J(x)} J(z) \right) = \delta^{(4)}(x-y)J(z) + \delta^{(4)}(x-z)J(y)$$