

Predictions for experiments: cross sections and decay rates
From QFT 1, p. 144

$$S_{ji} = \langle \text{out } \{k_j\} | \{p_i\} \text{in} \rangle \quad (11.48) \quad (1)$$

is the probability amplitude for a transition from $|\{p_i\}\rangle$
to $|\{k_j\}\rangle$

S-matrix-or scattering matrix element.*

The scattering probability is

$$P_{ji} = P(\{p_i\} \rightarrow \{k_j\}) = |S_{ji}|^2 \quad (2)$$

The S-matrix is unitary

$$(SS^\dagger)_{ij} = S_{ik} S_{kj}^\dagger = \delta_{ij} \quad (3)$$

One writes often

$$S = 1 + iT \quad (4)$$

where T is called the transition matrix and includes
non-diagonal processes. Note that

$$1 = SS^\dagger = (1 + iT)(1 - iT^\dagger) = 1 + i(T - T^\dagger) + TT^\dagger$$

$$2\text{Im } T = TT^\dagger \quad (5)$$

(important phenomenologically !!)

The P_{ij} are of interest experimentally, however not
directly. What is measured are cross-sections and
other rates.

* S_{ji} are the matrix elements of an operator \hat{S} that acts on
states. One sets $\langle \{k_j\} | \hat{S} | \{p_i\} \rangle = \langle \text{out } \{k_j\} | \{p_i\} \text{in} \rangle = S_{ji}$

Relating the S-matrix to calculable quantities

In QFT I, p. 150, it was shown that

$$S_{ji} = \langle \text{out } \{k_j\} | \{p_i\} \text{ in} \rangle \sim \mathcal{I}(\dots) \tilde{G}(\{k_j\}, \{p_i\}) \quad (6)$$

$$(11.76)$$

which shows that in order to get S , we need \tilde{G} (Green's functions). From the considerations in 12.6 (QFT I, p. 167) we find that

$$\tilde{G}(\{k_j\}, \{p_i\}) \sim (2\pi)^4 \delta^4(\sum k_j - \sum p_i) F_{ji} \quad (7)$$

(overall momentum conservation).

Therefore it is reasonable to equally write $S \sim (2\pi)^4 \dots$ and one sets

$$S_{ji} = (2\pi)^4 \delta^4(\sum k_j - \sum p_i) i M_{ji} \quad (9)$$

M is called invariant matrix element and is calculated from F_{ji} in (7) using Feynman rules.

Example: From QFT I, p. 169 one gets

$$G_d = \frac{(-i\lambda) (i)^4 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}{\mathcal{I}_i(p_i^2 - m^2 + i\epsilon)} \quad (10)$$

$$(12.79)$$

From (11.76) one has for the case d

$$S_{ji} = \mathcal{I}_i \left[\frac{(p_i^2 - m^2)}{i\sqrt{z}} \right] G_d \quad (11.76)$$

$$(11)$$

This tells us that

$$S_{ji} = (-i\lambda) (2\pi)^4 \delta^4(\dots) \quad (12)$$

Comparison of (9) and (12):

$$iM_{ji} = -i\lambda \quad M = -\lambda \quad (13)$$

In this way one can calculate any invariant matrix element. Examples follow.

Relating P_{ji} to Observables

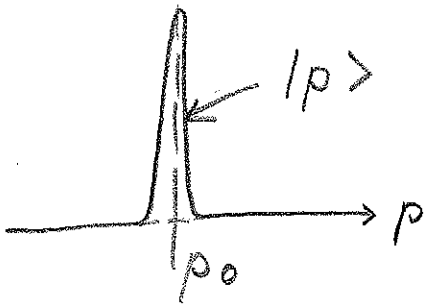
a) A mathematical problem

When calculating $P \sim |S|^2$ and using (9), one finds $(\delta^4)^2$ which is undefined mathematically. (one power of δ^4 is ok if we can integrate over momenta).

The problem stems from the fact that we consider only sharp momenta states $|p_0\rangle \sim \delta(p-p_0)$. When

looking at the norm, $\int dp | |p_0\rangle |^2$

$= 1$, we recognize that the value at p_0 must be infinite, and this leads to the problem.



As usual, the experimental set-up

resolves the problem. In reality, particles have a momentum spread, that is, they come in so called wave packets.

We write therefore for a state

$$|q\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k) |k\rangle \quad (14)$$

To understand the factors, remember (QFT I, p. 45)

$$|k\rangle = a_{\mathbf{k}}^+ |0\rangle \quad \text{and} \quad [a_{\mathbf{k}_1}, a_{\mathbf{k}_2}^+] = (2\pi)^3 (2E_{\mathbf{k}_1}) \delta^3(\mathbf{k}_1 - \mathbf{k}_2)$$

This yields

$$\begin{aligned} \langle k_1 | k_2 \rangle &= \langle 0 | a_{k_1} a_{k_2}^\dagger | 0 \rangle = \langle 0 | a_{k_1} a_{k_2}^\dagger - a_{k_2}^\dagger a_{k_1} + a_{k_2}^\dagger a_{k_1} | 0 \rangle \\ &= \langle 0 | (2\pi)^3 2E_{k_1} \delta(k_1 - k_2) | 0 \rangle = (2\pi)^3 2E_{k_1} \delta(k_1 - k_2) \end{aligned}$$

From this

$$\begin{aligned} \langle \phi | \phi \rangle &= \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \langle k | q \rangle \frac{\phi^*(k) \phi(q)}{\sqrt{2E_k 2E_q}} \\ &= \int \frac{d^3k}{(2\pi)^3} |\phi(k)|^2 \end{aligned} \tag{15}$$

Thus if $\langle \phi | \phi \rangle = 1$, then the "sum" (integral) of all probabilities $|\phi(k)|^2$ is 1, as desired.

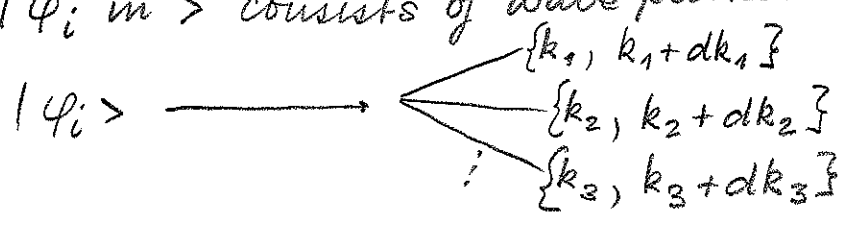
Using (+) for all states, we can set

$$\begin{aligned} \langle \text{out } \varphi_j | \varphi_i \text{ in} \rangle &= \tag{16} \\ \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \dots \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{\phi(p_1)}{\sqrt{2E_{p_1}}} \dots \frac{\phi(k_n)}{\sqrt{2E_{k_n}}} \langle \text{out } \{k_j\} | \{p_i\} \text{ in} \rangle \end{aligned}$$

We can simplify by taking the final state to consist of n particles with momenta in small regions d^3k_1, d^3k_2, \dots . Then the probability to go from the state $|\varphi_i \text{ in} \rangle$ to the final state is

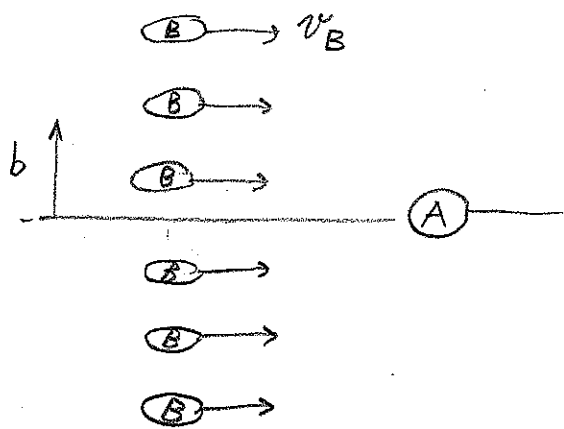
$$dP_{ji} = \prod_{a=1}^n \left(\frac{d^3k_a}{(2\pi)^3} \frac{1}{2E_a} \right) |\langle \text{out } \{k_j\} | \varphi_i \text{ in} \rangle|^2 \tag{17}$$

where $|\varphi_i \text{ in} \rangle$ consists of wave packets as in (+).



In the physical situation we may have a target particle A on which many incident particles B impinge. The initial state is (see comments)

$$\sim \int \frac{d^3p_A}{(2\pi)^3} \frac{d^3p_B}{\sqrt{2E_A 2E_B}} \phi_A \phi_B e^{-i b p_B} | \{p_i\} \text{ in} \rangle \tag{18}$$



The number N of scatterings is

$$N = \int d^2b n_B P(b) \quad (19)$$

Where n_B is the number of B particles per unit area

The cross section σ is defined by

$$N = \rho_B l_B \rho_A l_A \cdot F \cdot \sigma \quad (20)$$

where F is the common area to the B bunches and the target, ρ_B and l_B the density and length of the B bunches and likewise for A .

If the densities are constant over the area F , then $N_A = \rho_A l_A F$, etc. and

$$N = N_A \cdot N_B \frac{\sigma}{F} \quad (21)$$

Comparison of (xx) and (xxx) gives

$$\sigma = \frac{N \cdot F}{N_A N_B} = \frac{N}{n_B} \frac{1}{N_A} = \frac{\int d^2b n_B P(b)}{n_B \cdot N_A} \quad (22)$$

And if $N_A = 1$

$$\sigma = \int d^2b n_B P(b)$$

From this one can get

$$d\sigma = \frac{\pi}{a} \left(\frac{d^3 k_a}{(2\pi)^3 2E_a} \right) \int d^2b \pi \left(\int \frac{d^3 p_i \phi(p_i)}{(2\pi)^3 \sqrt{E_i} 2} \int \frac{d^3 \bar{p}_i \bar{\phi}(\bar{p}_i)}{(2\pi)^3 \sqrt{E_i} 2} \right) e^{i b (\bar{p}_B - p_B)} \langle \text{out } \{k_j\} | \{p_i\} \text{ in} \rangle \langle \text{out } \{k_j\} | \{\bar{p}_i\} \text{ in} \rangle^* \quad (23)$$

We now rewrite the matrix elements $\langle \dots \rangle$ using (9)*

as

$$\langle \dots \rangle = i M (2\pi)^4 \delta^4(\sum k_j - \sum p_i) \tag{24}$$

$$\langle \dots \rangle^* = -i M (2\pi)^4 \delta^4(\sum k_j - \sum \bar{p}_i) \tag{25}$$

Collecting terms in various directions:

$$\int d^2b e^{ib(\bar{p}_B - p_B)} = (2\pi)^2 \delta^2(\bar{p}_B^\perp - p_B^\perp) \tag{26}$$

$$\int \frac{d\bar{p}_A d\bar{p}_B}{(2\pi)^6 2\bar{E}_A 2\bar{E}_B} \delta(\bar{p}_A + \bar{p}_B - \sum k_j^z) \delta(\bar{E}_A + \bar{E}_B - \sum E_j) \tag{27}$$

(\bar{p}_A, \bar{p}_B go along z)

$$= \int d\bar{p}_A \delta(\sqrt{\bar{p}_A^2 + m_A^2} + \sqrt{\bar{p}_B^2 + m_B^2} - \sum E_j) \Big|_{k_B = \sum k_j - p_A}$$

$$= \frac{1}{\left| \frac{\bar{p}_A}{E_A} - \frac{\bar{p}_B}{E_B} \right|} = \frac{1}{|v_A - v_B|} \tag{28}$$

where $|v_A - v_B|$ is the relative velocity in the lab frame

One can re-express this factor (see below)

We return to (23). Since the ϕ are centered around p_A, p_B , we can essentially take out all smooth functions of these arguments. Then

$$dG = \pi \left(\frac{d^3k_a}{(2\pi)^3 2E_a} \right) \frac{|M|^2}{2E_a 2E_B |v_A - v_B|} \int \frac{d^3p_A d^3p_B}{(2\pi)^3 (2\pi)^3} |\phi(p_A)|^2 |\phi(p_B)|^2 (2\pi)^4 \delta(p_A + p_B - \sum k_j) \tag{29}$$

$$\int \delta(\sqrt{\bar{p}_A^2 + m_A^2} - \dots) = \frac{1}{\frac{p_A}{E_A} \dots}$$

In the next step, we take $p_A + p_B$ essentially fixed to the experimental value (this is justified, if the final momenta are measured with less precision than the variation of $p_A + p_B$). Then the δ -function goes out of the integral and with (15) we have

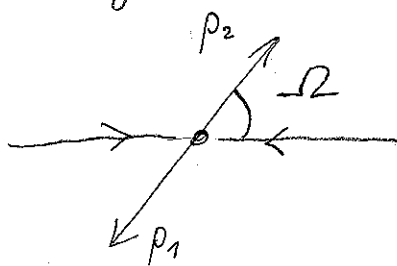
$$dG = \frac{\pi \left(\frac{d^3 k_j}{(2\pi)^3} \frac{1}{2E_j} \right)}{2E_A 2E_B |v_A - v_B|} |M|^2 (2\pi)^4 \delta^4(p_A + p_B - \sum p_f) \quad (30)$$

This is the final expression. It has the necessary invariance properties (under Lorentz-transformation).

For one $\left(\frac{d^3 k}{2E} \right)$ is invariant. But also $E_A E_B |v_A - v_B|$ is invariant under boost in z -direction and can be written (with proper use!) as

$$E_A E_B |v_A - v_B| = \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2} \quad (31)$$

For two particles in the final state, one can go to the center-of-mass frame



$$\frac{dG}{d\Omega} = \frac{1}{2 \cdot 2E_A E_B |v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4E_{CM}} |M|^2 \quad (32)$$

A similar formula holds for the decays of a single particle. One finds

$$d\Gamma = \frac{1}{2M_A} \left(\pi \frac{d^3 k_j}{(2\pi)^3} \frac{1}{2E_j} \right) |M(A \rightarrow \{k_j\})|^2 (2\pi)^4 \delta^4(p_A - \sum k_i)$$

(see Peskin/Schroeder, p. 107 etc)

Comments

The formula (xxx) for N can be understood as follows. $(\rho_B \ell_B)$, $(\rho_A \ell_A)$ are the numbers of particles per unit area. In the area F there are $\rho_B \ell_B F$ particles B (or $\rho_A \ell_A F$ particles A). The naive product $(\rho_A \ell_A F)(\rho_B \ell_B F)$ for N is wrong because this would mean that all particles scatter. We can correct by adding a factor ϵ for the probability. Then we can call (ϵF) the cross section, $\sigma = (\epsilon F)$. The reason we calculate (measure) σ rather than ϵ is that in experiments only the face F of the bunch counts, and not its length ()

The factor e^{-ibp_B} in (18) comes because the wave packet B is centered at a distance b .

To see this: The wave packets are at $x = vt e_x$. $\psi(x) \sim \int dp e^{-ipx} f(p)$. If $x = vt e_x + b$, the extra factor e^{-ipb} must be added.

Eq. (23). The 4-velocity v_μ is $(\gamma, \gamma \vec{v})$ ($c=1$). Then

$p_\mu = m v_\mu$, $p_0 = E = m\gamma$. Thus $\vec{p}/E = \vec{v}$.