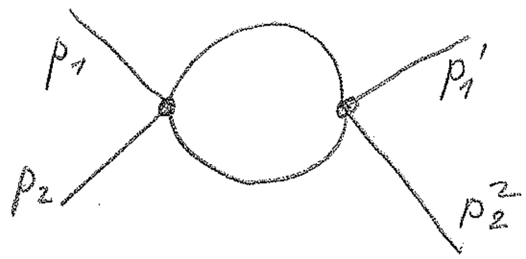
Consider, say, $\varphi\varphi \to \varphi\varphi$ scattering in $\frac{\lambda}{4!}\varphi^4$. The loop yields



$$A = g - \frac{g^2}{32\pi^2}\int_0^1 dx\left(\log\alpha_s + \log\alpha_t + \log\alpha_u - 3\right)$$

Where

$$\alpha_r = \frac{\Lambda^2}{m^2 - r\,x(1-x)} \qquad r = s, t, u\,,\quad \Lambda = \text{cutoff}$$

$$s = -(p_1+p_2)^2,\quad t = -(p_1-p_1')^2,\quad u = -(p_1-p_2')^2$$

$$s+t+u = 4m^2 \quad (\text{on shell})$$

We can remove the divergence by setting

$$g_R = A(s=t=u=0)$$

$$= g - \frac{3g^2}{32\pi^2}\left(\log\frac{\Lambda^2}{M^2} - 1\right)$$

and then

$$A = g_R + \frac{g_R^2}{32\pi^2}\int dx\left(\log\alpha_s' + \log\alpha_t' + \log\alpha_u'\right) + O(g_R^3)$$

$$\alpha_r' = \frac{m^2 - r\,x(1-x)}{m^2}$$

Instead of fixing g_R at $s=t=u=0$ or another fixed point of s, t, u, we can use a so called sliding scale μ, defined by $A(s=t=u=-\mu^2) \equiv g_\mu$. Then

$$g_\mu = g_R + \frac{3g_R^2}{32\pi^2}\int dx \log\left(1 + \frac{\mu^2 x(1-x)}{m^2}\right) + \dots$$

$$= g - \frac{3g^2}{32\pi^2}\int dx\left(\log\frac{\Lambda^2}{m^2+\mu^2 x(1-x)} - 1\right) \qquad (\mu R)$$

$$A = g_\mu - \frac{g_\mu^2}{32\pi^2}\int dx \left(\log\left(\frac{m^2+\mu^2 x(1-x)}{m^2+s\,x(1-x)}\right)\cdots\right)$$

for $m \to 0$, $s,t,u \to -\infty$ the form with g_R is singular, there are large logs $\log(s/m^2)$ etc. On the other hand, if g_μ is used, there is no problem and the corrections to $A = g_\mu$ are small. However if $\mu^2 \gg m^2$, g_μ is very different from g_R and the corrections are large (see (μR)); we need $\frac{3g_R}{32\pi^2}\log\frac{\mu^2}{m^2} \ll 1$.

The idea is now to find a formula that gives the all order relation between g_μ and g_R.

Consider $g_{\mu'}$ and g_μ. We have in lowest order (μR)

$$g_{\mu'} = \underbrace{g_\mu - \frac{3g_\mu^2}{32\pi^2}\int dx \log\left(\frac{m^2+\mu^2 x(1-x)}{m^2+\mu'^2 x(1-x)}\right)}_{g_\mu\left(1-\frac{3g_\mu}{32\pi}\int\log\right)} + O(g_\mu^3) \qquad (B)$$

We also see that in general (all orders)

$$g_{\mu'} = G\left(g_\mu, \frac{\mu'}{\mu}, \frac{m}{\mu}\right)$$

for dimensional reasons. This gives

$$\mu' \frac{d}{d\mu'} g_{\mu'} = \frac{\partial}{\partial z} G\left(g_\mu, z, \frac{m}{\mu}\right)$$

Set $\mu' = \mu$ we write

$$\mu \frac{d}{d\mu} g_\mu = \frac{\partial}{\partial z}\left(G(g_\mu, z, \frac{m}{\mu}\right)_{z=1} \equiv \beta(g_\mu, \frac{m}{\mu})$$

For $m \to 0$ ($\mu^2 \gg m^2$) this is

$$\mu \frac{d}{d\mu} g_\mu = \beta(g_\mu, 0) \equiv \beta(g_\mu)$$

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This gives

$$\frac{d g_\mu}{d \log\mu} = \beta(g_\mu) \to d\log\mu = \frac{dg_\mu}{\beta(g_\mu)}$$

$$\mathrm{Log}\left(\frac{\mu}{M}\right) = \int_{g_M}^{g_\mu} \frac{dg}{\beta(g)} \qquad (\beta \neq 0! \text{ needed})$$

where M is such that g_M can be obtained from g_R reliably and μ is a scale that corresponds to s, t, u.

From (B) we have

$$\beta = \frac{3 g_\mu^2}{16\pi^2} \int dx \frac{\mu^2 x(1-x)}{m^2 + \mu^2 x(1-x)} + \cdots$$

$$\beta(g_\mu) = \frac{3 g_\mu^2}{16\pi^2}$$

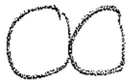
Then $\displaystyle \frac{d g_\mu}{d\log\mu} = \frac{3 g_\mu^2}{16\pi^2} \to g_\mu = -\frac{16\pi^2}{3\log(\mu/M)}$

$(g_M \gg g_\mu)$

Setting $g_M \sim g_R + \frac{3g_R^2}{16\pi^2} \log \frac{\mu}{M}$ from above

$$M \sim m \exp \frac{16\pi^2}{3g_R}$$

$$g_\mu = \frac{g_R}{1 - \frac{3g_R}{16\pi^2} \log \frac{\mu}{m}}$$

Sums  + ... (leading log)