

**Exercise 13.1 The Partition Function in  $\lambda\phi^4$  Theory**

The exercise below consists of parts of chapters 1 and 3 of [Kap89].

1. Our starting point is the action

$$S = -\frac{1}{2} \int_0^\beta d\tau \int d^3x \left[ \left( \frac{\partial\phi}{\partial\tau} \right)^2 + (\nabla\phi)^2 + m^2\phi^2 \right]$$

which we rewrite by partial integration, the boundary terms vanish because  $\phi$  is assumed to be periodic in  $\tau$  and vanishing at spatial infinity, this gives us the form

$$S = -\frac{1}{2} \int d\tau \int d^3x \phi [-\partial_\tau^2 - \Delta + m^2] \phi.$$

We expand the field in Fourier modes according to

$$\phi(\mathbf{x}, \tau) = \left( \frac{\beta}{V} \right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} e^{i(\mathbf{p}\mathbf{x} + \omega_n\tau)} \phi_n(\mathbf{p}).$$

The integration over  $\mathbf{x}$  and  $\tau$  results in  $V\delta(\mathbf{p} + \mathbf{p}')$  and  $\beta\delta(\omega_n + \omega_m)$  respectively. We remark that the reality condition  $\phi = \phi^*$  means  $\phi_{-n}(-\mathbf{p}) = \phi_n(\mathbf{p})^*$ . We have rewritten the action in the form

$$S = -\frac{1}{2}\beta^2 \sum_n \sum_{\mathbf{p}} [\omega_n^2 + \mathbf{p}^2 + m^2] |\phi_n(\mathbf{p})|^2.$$

We insert the action back into the partition function, ignoring prefactors of  $Z$  because these are irrelevant for thermodynamics:

$$Z \propto \int \mathcal{D}\phi \prod_n \prod_{\mathbf{p}} \exp \left[ -\frac{1}{2}\beta^2 (\omega_n^2 + \mathbf{p}^2 + m^2) |\phi_n(\mathbf{p})|^2 \right].$$

We do the integration over field configurations, the phases of the  $\phi_n(\mathbf{p})$  do give rise to an overall factor, for the integrations over the absolute values  $|\phi_n(\mathbf{p})|$  we insert  $\int \exp(-1/2 a x^2) dx \propto 1/\sqrt{a}$ :

$$Z \propto \prod_n \prod_{\mathbf{p}} [\beta^2 (\omega_n^2 + \mathbf{p}^2 + m^2)]^{-\frac{1}{2}}.$$

Of course, this is nothing else but the well-known functional determinant:

$$Z \propto (\det [-\partial_\tau^2 - \Delta + m^2])^{-\frac{1}{2}}.$$

We evaluate further the  $\beta$ -dependence, starting from

$$\ln Z = -\frac{1}{2} \sum_n \sum_{\mathbf{p}} \ln [\beta^2 (\omega_n^2 + \mathbf{p}^2 + m^2)].$$

We rewrite this in a form which enables us to do the summation over  $n$ . We abbreviate  $\omega^2 = \mathbf{p}^2 + m^2$  and insert

$$\ln [(2\pi n)^2 + (\beta\omega)^2] = \int_1^{(\beta\omega)^2} \frac{d(x^2)}{(2\pi n)^2 + x^2} + c$$

into the partition function, omitting the constant:

$$\begin{aligned} \ln Z &= -\frac{1}{2} \sum_n \sum_{\mathbf{p}} \int_1^{(\beta\omega)^2} \frac{d(x^2)}{(2\pi n)^2 + x^2} \\ &= -\frac{1}{2} \sum_{\mathbf{p}} \int_1^{(\beta\omega)^2} \frac{1}{2} \frac{\coth(x/2)}{x} d(x^2) \\ &= -\frac{1}{2} \sum_{\mathbf{p}} \int_1^{\beta\omega} \coth(x/2) dx \\ &= -\sum_{\mathbf{p}} \ln \sinh(\beta\omega/2) \end{aligned}$$

where we have inserted  $\sum_{n=-\infty}^{\infty} 1/(n^2 + x^2) = \pi/a \coth(\pi a)^1$  and  $\int \coth(x) dx = \ln \sinh(x)$  and we have omitted the  $\beta$ -independent lower integration boundary.

We continue by taking the continuum limit of the Fourier transform  $\sum_{\mathbf{p}} \rightarrow V/(2\pi)^3 \int d^3p$  and we insert  $\ln \sinh(x) = 1/2 + x/2 + \ln(1 - \exp(-2x))$ :

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[ -\frac{1}{2} - \frac{\beta\omega}{2} - \ln(1 - e^{-\beta\omega}) \right].$$

The constant inside the square bracket is omitted because it is  $\beta$ -independent, the term proportionate to  $\beta$  is the (highly UV-divergent) zero-point energy which we need to subtract anyway because we want  $P = 0$  in the limit  $\beta \rightarrow \infty$ . We consider the massless limit  $\omega = |\mathbf{p}|$  to arrive at

$$\begin{aligned} \ln Z &= V \int \frac{d^3p}{(2\pi)^3} (4\pi)(-1) \int d\omega \omega^2 \ln(1 - e^{-\beta\omega}) \\ &= V \frac{\pi^2}{90\beta^3} \end{aligned}$$

from which we have  $P = \beta^{-1} \partial_V \ln Z = \beta^{-4} \pi^2/90$ .

---

<sup>1</sup>We use the residue theorem for this, the function  $\pi i \coth(\pi i x)$  has residue one for  $x \in \mathbb{Z}$ . Therefore for  $ia \notin \mathbb{Z}$  the function  $\pi i \coth(\pi i x)/[(x + ia)(x - ia)]$  has residue  $1/(x^2 + a^2)$  for  $x \in \mathbb{Z}$ . We consider the integration along the contour  $C : z(\phi) = R \exp(i\phi)$  which vanishes as  $R \rightarrow \infty$ . From

$$0 = \lim_{R \rightarrow \infty} \int_C \pi i \coth(\pi i z)/[(z + ia)(z - ia)] dz = \sum_{n=-\infty}^{\infty} \frac{1}{a^2 + n^2} + \pi i \coth(-\pi a) \frac{1}{2ia} + \pi i \coth(\pi a) \frac{1}{-2ia}$$

we get the result inserted above. To check  $|\coth(R \exp i\phi)| \leq c$  we insert

$$|\coth(R e^{i\phi})|^2 = \left| \frac{e^{R(\cos \phi + i \sin \phi)} + e^{-R(\cos \phi + i \sin \phi)}}{e^{R(\cos \phi + i \sin \phi)} - e^{-R(\cos \phi + i \sin \phi)}} \right|^2 = \frac{1 + \frac{\cos(2R \sin \phi)}{\cosh(2R \cos \phi)}}{1 - \frac{\cos(2R \sin \phi)}{\cosh(2R \cos \phi)}}.$$

Although the function  $\cos(2R \sin(\phi))/\cosh(2R \cos(\phi))$  is equal to one at  $\phi = \pi/2 + n_1\pi, R = n_2\pi$ , we can choose the sequence of radii  $R_n$  such that  $\cos(2R_n \sin(\phi))/\cosh(2R_n \cos(\phi)) \leq c' < 1$  for all  $n$ .

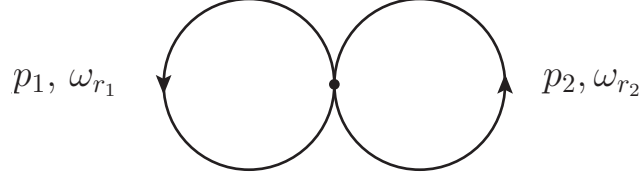


Figure 1: The two-loop vacuum bubble diagram of  $\lambda\phi^4$  theory. Its symmetry factor is 3.

2. We condense the notation first,  $\sum_n$  is used for the sum over  $n$  and  $\mathbf{p}$ ,  $\exp(i\omega_n\tau)$  should be understood as  $\exp(i\omega_n\tau + i\mathbf{p}\mathbf{x})$  in consequence. Inserting the Fourier transforms we have therefore

$$\ln Z_1 = \frac{-\lambda \sum_{n_1, \dots, n_4} \frac{\beta^2}{V^2} \int d\tau \int d^3x \int \prod_k d\phi_k e^{-\frac{1}{2}\beta^2 A_k |\phi_k|^2} (\prod_{i=1}^4 \phi_{n_i}) e^{i(\omega_{n_1} + \dots + \omega_{n_4})\tau}}{\prod_k d\phi_k e^{-\frac{1}{2}\beta^2 A_k |\phi_k|^2}}.$$

with  $A_k = \omega_k^2 + \mathbf{p}^2 + m^2$ . The integral over  $\phi_k$  vanishes by antisymmetry under  $\phi_k \rightarrow -\phi_k$  unless the term is of the form  $\phi_{r_1} \phi_{-r_1} \phi_{r_2} \phi_{-r_2} = |\phi_{r_1}|^2 |\phi_{r_2}|^2$ . In the four-fold sum every term  $|\phi_{r_1}|^4$  shows up once, every term  $|\phi_{r_1}|^2 |\phi_{r_2}|^2$  with  $r_1 \neq r_2$  shows up three times ( $r_1 = -r_2, r_1 = -r_3, r_1 = -r_4$ ). We insert

$$\begin{aligned} \int dx \exp^{-\frac{1}{2}ax^2} dx &= \sqrt{2\pi a}^{-\frac{1}{2}}, \\ \int dx \exp^{-\frac{1}{2}ax^2} x^2 dx &= \sqrt{2\pi a}^{-\frac{3}{2}}, \\ \int dx \exp^{-\frac{1}{2}ax^2} x^4 dx &= 3\sqrt{2\pi a}^{-\frac{5}{2}} \end{aligned}$$

for every  $d\phi_m$ . All the terms  $\sqrt{2\pi a}^{-1/2}$  in the numerator are cancelled by equal terms in the denominator. We are left with a double sum

$$\ln Z_1 = -3\lambda \sum_{r_1, \mathbf{p}_1, r_2, \mathbf{p}_2} \frac{\beta^2}{V^2} (\beta^2 [\omega_{r_1}^2 + \mathbf{p}_1^2 + m^2])^{-1} (\beta^2 [\omega_{r_2}^2 + \mathbf{p}_2^2 + m^2])^{-1} \beta V$$

where the trailing  $\beta V$  arises from  $\int d^3x d\tau \delta(\mathbf{x}) \delta(\tau)$  and we have written the combinatorial factor 3 explicitly. We take the continuum limit of the Fourier transform as above:

$$\ln Z_1 = -3\lambda\beta V \left[ \beta^{-1} \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2} \right]^2. \quad (1)$$

We recognise the expression above as the two-loop integral in figure 1 with  $(\omega_n^2 + \mathbf{p}^2 + m^2)^{-1}$  as the propagator associated with an internal line.

3. We evaluate  $\ln Z_1$  for  $m = 0$ , we have therefore  $\omega = |\mathbf{p}|$ . First we do the sum in the propagator:

$$\sum_n \frac{1}{\omega_n^2 + \omega^2} = \frac{1}{2\omega} \coth\left(\frac{\beta\omega}{2}\right).$$

We evaluate the integral over Fourier modes with a cutoff:

$$\begin{aligned} \int_{|\mathbf{p}| \leq \Lambda} \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} \coth\left(\frac{\beta\omega}{2}\right) &= \frac{1}{4\pi^2} \int_0^\Lambda \omega \coth\left(\frac{\beta\omega}{2}\right) \\ &= \frac{1}{4\pi^2} \left[ \frac{\pi^2}{3\beta^2} + \frac{\Lambda^2}{2} + \frac{2\Lambda \ln(1 - e^{-\beta\Lambda})}{\beta} - \frac{2\text{Li}_2(e^{-\beta\Lambda})}{\beta^2} \right]. \end{aligned}$$

The second term in the expression above is  $\beta$ -independent and is therefore omitted, the other two  $\Lambda$ -dependent terms vanish in the limit  $\Lambda \rightarrow \infty$ . If we insert this back into (1), we arrive at

$$P_1 = \beta^{-1} \partial_V \ln Z_1 = -3\lambda \left( \frac{1}{12\beta^2} \right)^2 = -\frac{\lambda}{48} \beta^{-4}.$$

## References

- [Kap89] Joseph I. Kapusta. *Finite-temperature field theory*. Cambridge University Press, 1989.