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Advanced Field Theory
Solution 12

Spring 2010
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Exercise 12.1 Derrick's Theorem

We have with $\phi_i' = \phi_i(\mathbf{x}/\alpha)$:

$$\begin{aligned}
T' &= \frac{1}{2}\int \mathrm{d}^D x \frac{1}{\alpha^2}\left(\nabla\phi_i\left(\frac{\mathbf{x}}{\alpha}\right)\right) \\
&= \frac{1}{2}\int \mathrm{d}^D\,(x\alpha)\,\alpha^{D-2}\left(\nabla\phi_i\left(\frac{\mathbf{x}}{\alpha}\right)\right)^2 \\
&= \alpha^{D-2}T
\end{aligned}$$

and for the potential energy

$$U' = \int \mathrm{d}^D x V\left(\phi_i\left(\frac{\mathbf{x}}{\alpha}\right)\right) = \alpha^D U.$$

We need $\alpha = 1$ to be a local minimum of the energy because this is nothing but the condition that the soliton should be stable against growing or shrinking.

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}E\bigg|_{\alpha=1} = (D-2)T + DU$$

which we set to zero to get

$$(2-D)T = DU$$

which is in contradiction to $T > 0$ and $U > 0$ for $D \geq 2$.
We can also argue with the sign of the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}E = (D-2)\alpha^{D-3}T + D\alpha^{D-1}U.$$

For $D \geq 2$ this is a positive function for all $\alpha > 0$, from this we conclude that the soliton spectrum extends down to $E = 0$ which is approached by spreading out the soliton more and more. Therefore by taking $\alpha \to 0$ we transform the soliton into the trivial vacuum ground state. This is in contradiction to the fact that ϕ_i was assumed to be a soliton solution.