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Advanced Field Theory
Solution 11

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Exercise 11.1 Magnetic monopoles

We insert the ansatz

$$\phi_a = \frac{H(\xi)}{\xi}\eta\frac{x_a}{r}, \qquad A_i^a = \frac{\epsilon_{aij}x_j}{gr^2}\left(K(\xi)-1\right)$$

1. We remark

$$\partial_i\frac{x_a}{r} = \frac{\delta_{ai}}{r} - \frac{x_i x_a}{r^3}, \qquad \partial_i\xi = g\eta\frac{x_i}{r} = g^2\eta^2\frac{x_i}{\xi}, \qquad \partial_i r^n = n x_i r^{n-2}.$$

   We calculate the parts in order:

$$\begin{aligned}\partial_i\phi_a &= \partial_i\left(\frac{H}{\xi}\eta\frac{x_a}{r}\right)\\ &= \eta\left(\frac{x_i x_a}{r^3}\xi\left[\frac{H'}{\xi}-\frac{H}{\xi^2}\right]+\frac{H}{\xi}\left[\frac{\delta_{ai}}{r}-\frac{x_i x_a}{r^3}\right]\right)\\ &= \eta\left(\frac{x_i x_a}{r^3}\left[H'-2\frac{H}{\xi}\right]+\frac{\delta_{ai}}{r}\frac{H}{\xi}\right)\end{aligned}$$

$$\begin{aligned}-g\epsilon_{abc}A_i^b\phi_c &= -g\epsilon_{abc}\frac{\epsilon_{bik}x_k}{gr^2}(K-1)\frac{H}{\xi}\eta\frac{x_c}{r}\\ &= -\eta\frac{H}{\xi}\frac{1}{r^3}(K-1)(x_i x_a - r^2\delta_{ai})\\ &= \eta\left(\frac{x_i x_a}{r^3}\left[-\frac{H}{\xi}(K-1)\right]+\frac{\delta_{ai}}{r}\frac{H}{\xi}(K-1)\right)\end{aligned}$$

   adding up to

$$\begin{aligned}\partial_i\phi_a - g\epsilon_{abc}A_i^b\phi_c &= \eta\left(\frac{x_i x_a}{r^3}\left[H'-\frac{H}{\xi}-\frac{HK}{\xi}\right]+\frac{\delta_{ai}}{r}\frac{H}{\xi}K\right)\\ &= KH\frac{1}{gr^4}\left(r^2\delta_{ai}-x_i x_a\right)+\frac{x_i x_a}{gr^4}\left(\xi H'-H\right).\end{aligned}$$

2. For the square we need

$$(r^2\delta_{ai}-x_i x_a)(r^2\delta_{ai}-x_i x_a) = r^4\delta_{ii}+x_i x_i x_a x_a - 2r^2 x_i x_i = 2r^4$$

$$(r^3\delta_{ai}-x_i x_a)x_i x_a = r^4 - r^4 = 0$$

   and therefore we have for the kinetic term

$$\frac{1}{2}(D_\mu\phi)(D^\mu\phi) = \frac{-1}{2}(D_i\phi)(D_i\phi) = \frac{-1}{2}\frac{g^2\eta^4}{\xi^4}\left(2K^2H^2+(\xi H'-H)^2\right).$$

We proceed with the field strength term, we have

$$
\begin{aligned}
\partial_i A^a_j &= \frac{1}{g}\epsilon_{ajk}\partial_i\left(x_k\frac{1}{r^2}(K-1)\right)\\
&= \frac{1}{g}\epsilon_{ajk}\left(\delta_{ik}r^{-2}(K-1) - 2x_ix_kr^{-4}(K-1) + x_kr^{-2}K'g\eta\frac{x_i}{r}\right)\\
&= \frac{1}{g}\epsilon_{aji}\frac{K-1}{r^2} + \frac{1}{g}x_i\epsilon_{ajk}x_k\left(-2r^{-4}(K-1) + g\eta r^{-3}K'\right)\\
&= \frac{-1}{g}\frac{K-1}{r^2}\epsilon_{aij} + \frac{1}{g}x_i\epsilon_{ajk}x_kr^{-4}\left(\xi K' - 2(K-1)\right)
\end{aligned}
$$

and therefore

$$
\partial_i A^a_j - \partial_j A^a_i = \frac{-2}{g}\frac{K-1}{r^2}\epsilon_{aij} + \frac{1}{g}(x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k)r^{-4}(\xi K' - 2(K-1)).
$$

For the non-abelian part we have

$$
\begin{aligned}
\epsilon_{abc}A^b_iA^c_j &= \epsilon_{abc}\epsilon_{bik}x_k\epsilon_{cjm}x_m\frac{1}{g^2}\frac{1}{r^4}(K-1)^2\\
&= (\delta_{ci}\delta_{ak} - \delta_{ck}\delta_{ai})x_k\epsilon_{cjm}x_m\frac{1}{g^2}\frac{1}{r^4}(K-1)^2\\
&= x_a\epsilon_{ijm}x_m\frac{1}{g^2}\frac{1}{r^4}(K-1)^2
\end{aligned}
$$

giving us

$$
F^a_{ij} = \frac{-2}{g}\frac{K-1}{r^2}\epsilon_{aij} + \frac{1}{g}(x_i\epsilon_{ajk}x_k - x_j\epsilon_{aim}x_m)r^{-4}\left(\xi K' - 2(K-1)\right) - \frac{1}{g}\frac{(K-1)^2}{r^4}x_a\epsilon_{ijm}x_m.
$$

In preparation for the contraction, we calculate

$$
\begin{aligned}
\epsilon_{aij}\epsilon_{aij} &= 6 & x_i\epsilon_{ajk}x_kx_i\epsilon_{ajm}x_m &= 2r^4\\
x_i\epsilon_{ajk}x_kx_j\epsilon_{ajm}x_m &= 0 & (x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k)^2 &= 4r^4\\
x_ax_a\epsilon_{ijm}\epsilon_{ijk}x_mx_k &= 2r^4 & \epsilon_{aij}(x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k) &= -4r^2\\
\epsilon_{aij}x_a\epsilon_{ijm}x_m &= 2r^2 & (x_i\epsilon_{ajk}x_k - x_j\epsilon_{aik}x_k)x_a\epsilon_{ijm}x_m &= 0
\end{aligned}
$$

We insert this into $F^a_{ij}F^{ij,a} = F^a_{ij}F^a_{ij} = (F^a_{ij})^2$:

$$
\begin{aligned}
(F^a_{ij})^2 = &\frac{24}{g^2}\frac{(K-1)^2}{r^4} + \frac{1}{g^2}4r^{-4}(\xi K' - 2(K-1)^2)^2 + \frac{1}{g^2}\frac{(K-1)^4}{r^8}\cdot 2r^4\\
&- \frac{4}{g^2}(K-1)(\xi K' - 2(K-1))(-4r^2)r^{-6} + \frac{4}{g^2}\frac{(K-1)^3}{r^6}\cdot 2r^2\\
= &\frac{1}{g^2r^4}\left[8(K-1)^2 + 2(K-1)^4 + 8(K-1)^3 + 4(\xi K')^2\right]\\
= &\frac{1}{g^2r^4}\left[2(K^2-1)^2 + 4(\xi K')^2\right]
\end{aligned}
$$

$$
\frac{-1}{4}(F^a_{ij})^2 = \frac{-g^2\eta^4}{2\xi^4}\left[(K^2-1)^2 + 2(\xi K')^2\right]
$$

We do need the potential term as well, we have

$$
\frac{-\lambda}{8}\left(\phi^2 - \eta^2\right)^2 = \frac{-\lambda}{8}\frac{\eta^4}{\xi^4}\left(H^2 - \xi^2\right)^2.
$$

Now we add things up to arrive at the energy

$$E = -\int \mathrm{d}\mathbf{x}\,\mathcal{L} = -\int \frac{4\pi}{g^3\eta^3}\xi^2 \mathrm{d}\xi\,\mathcal{L} = \frac{4\pi}{g^3\eta^3}\int \xi^2 \mathrm{d}\xi \left(\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} - \frac{1}{2}D_\mu\phi D^\mu\phi + \frac{\lambda}{8}\left(\phi^2-\eta^2\right)^2\right)$$
$$= \frac{4\pi\eta}{g}\int \mathrm{d}\xi \frac{1}{\xi^2}\left(\frac{1}{2}(K^2-1)^2 + (\xi K')^2 + K^2H^2 + \frac{1}{2}(\xi H' - H)^2 + \frac{\lambda}{8g^2}(H^2-\xi^2)^2\right).$$

3. We assume $E[g] = \int f(g(x), g'(x))\mathrm{d}x$ and we recall

$$\begin{aligned}\frac{\delta E}{\delta g}[\phi] &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\int f(g+\epsilon\phi, g'+\epsilon\phi')\mathrm{d}x\bigg|_{\epsilon=0}\\ &= \int \frac{\partial f(g,g')}{\partial g}\phi + \frac{\partial f(g,g')}{\partial g'}\phi'\mathrm{d}x\\ &= \int \left(\frac{\partial f(g,g')}{\partial g} - \frac{\partial}{\partial x}\frac{\partial f(g,g')}{\partial g'}\right)\phi\,\mathrm{d}x\end{aligned}$$

where we have assumed that the boundary terms of the partial integration vanish. Therefore we get from the condition that the derivatives of the energy with respect to K and H should vanish:

$$\frac{\delta E}{\delta K} = 0 \Rightarrow K'' = \frac{1}{\xi^2}\left(K(K^2-1) + 2KH^2\right)$$
$$\frac{\delta E}{\delta H} = 0 \Rightarrow H'' - \frac{H'}{\xi} + \frac{H}{\xi^2} = \frac{2K^2H}{\xi^2} - \frac{H'}{\xi} + \frac{H}{\xi^2} + \frac{1}{\xi^2}\frac{\lambda}{2g^2}H(H^2-\xi^2).$$