

**Exercise 10.1 The soliton of  $\lambda\phi^4$  theory in 1 + 1 dimensions**

1. With

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi), \quad V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

the constant solutions are  $\phi(x) = \pm\sqrt{6}|m|/\sqrt{\lambda}$ , therefore we shift the potential to be

$$V'(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{3|m|^4}{2\lambda}.$$

2. The static soliton solutions are given by

$$\phi(x) = \frac{\sqrt{6}|m|}{\sqrt{\lambda}} \tanh\left(\frac{|m|}{\sqrt{2}}x\right).$$

3. We have

$$E = \int dx \frac{1}{2}(\partial_x\phi)^2 + V'(\phi)$$

we insert

$$\partial_x\phi = \frac{\sqrt{3}|m|^2}{\sqrt{\lambda}} \cosh^{-2}\left(\frac{|m|}{\sqrt{2}}x\right), \quad \frac{1}{2}(\partial_x\phi)^2 = \frac{3|m|^4}{2\lambda} \cosh^{-4}\left(\frac{|m|}{\sqrt{2}}x\right)$$

which we combine with

$$\begin{aligned} V'(\phi) &= \frac{-1}{2}|m|^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{3|m|^4}{2\lambda} \\ &= \frac{3|m|^4}{2\lambda} \left(-2 \tanh^2\left(\frac{|m|}{\sqrt{2}}x\right) + \tanh^4\left(\frac{|m|}{\sqrt{2}}x\right) + 1\right) \\ &= \frac{3|m|^4}{2\lambda} \cosh^{-4}\left(\frac{|m|}{\sqrt{2}}x\right) \end{aligned}$$

to have

$$E = \frac{3|m|^4}{2\lambda} \frac{\sqrt{2}}{|m|} \frac{8}{3} = \frac{4\sqrt{2}|m|^3}{\lambda}$$

4. We normalise

$$J^\mu = \frac{\sqrt{\lambda}}{2\sqrt{6}|m|} \epsilon^{\mu\nu} \partial_\nu\phi.$$

$\partial_\mu J^\mu$  vanishes due to the antisymmetry of  $\epsilon$ , the possible values are given by

$$\int J^0 = \frac{\sqrt{\lambda}}{2\sqrt{6}|m|} \int dx \partial_x\phi = \frac{\sqrt{\lambda}}{2\sqrt{6}|m|} \phi|_{-\infty}^{\infty} = \pm 1, 0$$

corresponding to the fact that there are two possible asymptotic values of  $\phi(x)$ .