

**Exercise 9.1 Pure gauge**

We use

$$\partial_\mu(\Lambda\Lambda^{-1}) = 0 = \Lambda\partial_\mu\Lambda^{-1} + (\partial_\mu\Lambda)\Lambda^{-1}$$

to show

$$\begin{aligned} [A_\mu, A_\nu] &= \frac{-1}{g^2} ((\partial_\mu\Lambda)\Lambda^{-1}(\partial_\nu\Lambda)\Lambda^{-1} - (\partial_\nu\Lambda)\Lambda^{-1}(\partial_\mu\Lambda)\Lambda^{-1}) \\ &= \frac{-1}{g^2} (-(\partial_\mu\Lambda)\Lambda^{-1}\Lambda(\partial_\nu\Lambda^{-1}) + (\partial_\nu\Lambda)\Lambda^{-1}\Lambda(\partial_\mu\Lambda^{-1})) \\ &= \frac{-1}{g^2} ((\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})) \end{aligned}$$

and we insert

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &= \frac{i}{g} (\partial_\mu ((\partial_\nu\Lambda)\Lambda^{-1}) - \partial_\nu ((\partial_\mu\Lambda)\Lambda^{-1})) \\ &= \frac{i}{g} ((\partial_\mu\partial_\nu\Lambda)\Lambda^{-1} + (\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\nu\partial_\mu\Lambda)\Lambda^{-1} - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})) \\ &= \frac{i}{g} ((\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})) \end{aligned}$$

giving us

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] = 0.$$

**Exercise 9.2 Instantons**

The instanton solution treated here was first reported in [BPST75], the form using  $\eta_{\mu\nu}^a$  is from [tH76] where the quantum effects of instantons were considered.

We have

$$\partial_\mu A_\nu = \frac{-4}{g} \frac{1}{(x^2 + \lambda^2)^2} x_\mu \eta_{\nu\kappa}^a x_\kappa - \frac{2}{g} \frac{1}{x^2 + \lambda^2} \eta_{\mu\nu}^a$$

which we insert to have

$$\partial_\mu A_\nu - \partial_\nu A_\mu = \frac{-4}{g} \frac{1}{(x^2 + \lambda^2)^2} x_\mu \eta_{\nu\kappa}^a x_\kappa + \frac{4}{g} \frac{1}{(x^2 + \lambda^2)^2} x_\nu \eta_{\mu\kappa}^a x_\kappa - \frac{4}{g} \frac{1}{x^2 + \lambda^2} \eta_{\mu\nu}^a.$$

We insert the contraction identity on the exercise sheet for the nonabelian part of the field strength tensor:

$$\begin{aligned} g\epsilon_{abc}A_\mu^b A_\nu^c &= \frac{4}{g} \frac{1}{(x^2 + \lambda^2)^2} \epsilon_{abc} \eta_{\mu\kappa}^b \eta_{\nu\rho}^c x_\kappa x_\rho \\ \epsilon_{abc} \eta_{\mu\kappa}^b \eta_{\nu\rho}^c x_\kappa x_\rho &= (\delta_{\mu\nu} \eta_{\kappa\rho}^a - \delta_{\mu\rho} \eta_{\kappa\nu}^a - \delta_{\kappa\nu} \eta_{\mu\rho}^a + \delta_{\kappa\rho} \eta_{\mu\nu}^a) x_\kappa x_\rho \\ &= x_\mu \eta_{\nu\kappa}^a x_\kappa - x_\nu \eta_{\mu\kappa}^a x_\kappa + (x^2 + \lambda^2) \eta_{\mu\nu}^a - \lambda^2 \eta_{\mu\nu}^a \end{aligned}$$

From this we see that in the sum all the terms proportionate to  $(x^2)^{-1}$  for large  $x$  cancel:

$$\partial_\mu A_\nu - \partial_\nu A_\mu + g\epsilon_{abc}A_\mu^b A_\nu^c = \frac{4}{g} \frac{-\lambda^2 \eta_{\mu\nu}^a}{(x^2 + \lambda^2)^2}.$$

We insert this field strength tensor into the action, using  $\eta_{\mu\nu}^a \eta_{\mu\nu}^a = 12$ :

$$\begin{aligned}\underline{S} &= \frac{48\lambda^4}{g^2} \int d^4x_E \frac{1}{(x^2 + \lambda^2)^4} \\ &= \frac{48\lambda^4}{g^2} \frac{\pi^2}{6\lambda^4} \\ &= \frac{8\pi^2}{g^2}\end{aligned}$$

where we have inserted the integration from assignment 1.

It should be noted that from the formula for the action, we can see immediately that the weight  $\exp(-\underline{S})$  of the instanton configuration has an essential singularity at  $g = 0$ , rendering a weak-coupling expansion of this expression useless.

## References

- [BPST75] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin. Pseudoparticle solutions of the yang-mills equations. *Physics Letters B*, 59(1):85 – 87, 1975.
- [tH76] G. 't Hooft. Computation of the quantum effects due to a four-dimensional pseudoparticle. *Phys. Rev. D*, 14(12):3432–3450, Dec 1976.