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Advanced Field Theory
Solution 9

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Exercise 9.1 Pure gauge

We use

$$\partial_\mu(\Lambda\Lambda^{-1}) = 0 = \Lambda\partial_\mu\Lambda^{-1} + (\partial_\mu\Lambda)\Lambda^{-1}$$

to show

$$\begin{aligned}
[A_\mu, A_\nu] &= \frac{-1}{g^2}\left((\partial_\mu\Lambda)\Lambda^{-1}(\partial_\nu\Lambda)\Lambda^{-1} - (\partial_\nu\Lambda)\Lambda^{-1}(\partial_\mu\Lambda)\Lambda^{-1}\right) \\
&= \frac{-1}{g^2}\left(-(\partial_\mu\Lambda)\Lambda^{-1}\Lambda(\partial_\nu\Lambda^{-1}) + (\partial_\nu\Lambda)\Lambda^{-1}\Lambda(\partial_\nu\Lambda^{-1})\right) \\
&= \frac{-1}{g^2}\left((\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})\right)
\end{aligned}$$

and we insert

$$\begin{aligned}
\partial_\mu A_\nu - \partial_\nu A_\mu &= \frac{i}{g}\left(\partial_\mu\left((\partial_\nu\Lambda)\Lambda^{-1}\right) - \partial_\nu\left((\partial_\mu\Lambda)\Lambda^{-1}\right)\right) \\
&= \frac{i}{g}\left((\partial_\mu\partial_\nu\Lambda)\Lambda^{-1} + (\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\nu\partial_\mu\Lambda)\Lambda^{-1} - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})\right) \\
&= \frac{i}{g}\left((\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})\right)
\end{aligned}$$

giving us

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu] = 0.$$

Exercise 9.2 Instantons

The instanton solution treated here was first reported in [BPST75], the form using $\eta^a_{\mu\nu}$ is from [tH76] where the quantum effects of instantons were considered.
We have

$$\partial_\mu A_\nu = \frac{-4}{g}\frac{1}{(x^2+\lambda^2)^2}x_\mu\eta^a_{\nu\kappa}x_\kappa - \frac{2}{g}\frac{1}{x^2+\lambda^2}\eta^a_{\mu\nu}$$

which we insert to have

$$\partial_\mu A_\nu - \partial_\nu A_\mu = \frac{-4}{g}\frac{1}{(x^2+\lambda^2)^2}x_\mu\eta^a_{\nu\kappa}x_\kappa + \frac{4}{g}\frac{1}{(x^2+\lambda^2)^2}x_\nu\eta^a_{\mu\kappa}x_\kappa - \frac{4}{g}\frac{1}{x^2+\lambda^2}\eta^a_{\mu\nu}.$$

We insert the contraction identity on the exercise sheet for the nonabelian part of the field strength tensor:

$$g\epsilon_{abc}A^b_\mu A^c_\nu = \frac{4}{g}\frac{1}{(x^2+\lambda^2)^2}\epsilon_{abc}\eta^b_{\mu\kappa}\eta^c_{\nu\rho}x_\kappa x_\rho$$

$$\begin{aligned}
\epsilon_{abc}\eta^b_{\mu\kappa}\eta^c_{\nu\rho}x_\kappa x_\rho &= \left(\delta_{\mu\nu}\eta^a_{\kappa\rho} - \delta_{\mu\rho}\eta^a_{\kappa\nu} - \delta_{\kappa\nu}\eta^a_{\mu\rho} + \delta_{\kappa\rho}\eta^a_{\mu\nu}\right)x_\kappa x_\rho \\
&= x_\mu\eta^a_{\nu\kappa}x_\kappa - x_\nu\eta^a_{\mu\kappa}x_\kappa + (x^2+\lambda^2)\eta^a_{\mu\nu} - \lambda^2\eta^a_{\mu\nu}
\end{aligned}$$

From this we see that in the sum all the terms proportionate to $(x^2)^{-1}$ for large x cancel:

$$\partial_\mu A_\nu - \partial_\nu A_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu = \frac{4}{g}\frac{-\lambda^2\eta^a_{\mu\nu}}{(x^2+\lambda^2)^2}.$$

We insert this field strength tensor into the action, using $\eta^a_{\mu\nu}\eta^a_{\mu\nu} = 12$:

$$\begin{aligned} \underline{S} &= \frac{48\lambda^4}{g^2} \int \mathrm{d}^4 x_E \frac{1}{(x^2+\lambda^2)^4} \\ &= \frac{48\lambda^4}{g^2} \frac{\pi^2}{6\lambda^4} \\ &= \frac{8\pi^2}{g^2} \end{aligned}$$

where we have inserted the integration from assignment 1.
It should be noted that from the formula for the action, we can see immediately that the weight $\exp(-\underline{S})$ of the instanton configuration has an essential singularity at $g = 0$, rendering a weak-coupling expansion of this expression useless.

References

[BPST75] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin. Pseudoparticle solutions of the yang-mills equations. *Physics Letters B*, 59(1):85 – 87, 1975.

[tH76] G. 't Hooft. Computation of the quantum effects due to a four-dimensional pseudoparticle. *Phys. Rev. D*, 14(12):3432–3450, Dec 1976.