Advanced Field Theory
Solution 8

Spring 2010
C. Anastasiou
T. Gehrmann

The derivation follows [VW84a] closely which references [VW84b] for the positivity of the fermion determinant. The conventions for QCD in Minkowski space follow [Wei95].
We proceed by the following strategy: Let $\mathcal{L}$ be the Lagrangian density of the theory under consideration, X is a hermitian, parity-odd operator (below we will take it to be $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$. We consider the theories described by the Lagrangians $\mathcal{L}_\lambda = \mathcal{L} + \lambda X$ with λ a real parameter. We have for the ground state energy

$$E_0(\lambda) = E_0(0) + \lambda \int \mathrm{d}^3x \, \langle X \rangle + \mathcal{O}\left(\lambda^2\right)$$

where $\langle X \rangle$ denotes a vacuum expectation value of X in the theory with $\lambda = 0$. If $\langle X \rangle$ is nonzero, then there are at least two vacua related by parity, one with vacuum expectation value $\langle X \rangle$, one with vacuum expectation value $-\langle X \rangle$. Therefore there is a vacuum state with $E_0(\lambda) < E_0(0)$ regardless of the sign of λ if $\langle X \rangle \neq 0$. We will show below that $E_0(\lambda \neq 0) > E_0(0)$ which implies by the argument above that X does not acquire a vacuum expectation value in the original theory with $\lambda = 0$.
We fix conventions as follows: we consider the QCD Lagrangian

$$\mathcal{L} = \frac{-1}{4} \operatorname{Tr} F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(\not{D} + M)\psi + \lambda F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad \not{D} = \gamma^\mu \partial_\mu + igA_\mu$$

with positive definite mass matrix M. Furthermore we have the metric $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, the gamma matrices satisfy $\gamma_0^\dagger = -\gamma_0$, $\gamma_k^\dagger = \gamma_k$, we fix $i\gamma^0 = \gamma^4 = \gamma_4$.
From the effective action we have

$$e^{-iTV_3E_0} = \int \mathcal{D}\phi e^{iS[\phi]}$$

where $T \cdot V_3$ is the spacetime volume, ϕ is used to denote all the fields of the theory and $S[\phi] = \int \mathrm{d}^4x \mathcal{L}(\phi)$ denotes the action.
We take T to mean $(\Delta x)^0$, therefore we do the analytic continuation onto the negative imaginary x^0-axis:

$$\begin{aligned}
x^0 &= -ix^4 = -ix_4 \\
x_0 &= ix_4 = ix^4 \\
\mathrm{d}^4x &= \mathrm{d}x^0\mathrm{d}^3\mathbf{x} = -i\mathrm{d}^4x_E \\
\partial^0 &= -i\partial^4 = -i\partial_4 \\
A^0 &= -iA^4 = -iA_4
\end{aligned}$$

in the analytic continuation of the Lagrangian we have

$$F_{\mu\nu}F^{\mu\nu} \to F^E_{\mu\nu}F^E_{\mu\nu}$$

because $F_{0k}F^{0k} \to (iF^E_{4k})(-iF^E_{4k})$. For the fermion part we have

$$\bar{\psi}(\gamma^\mu D_\mu + M)\psi \to \bar{\psi}(\gamma_\mu D_\mu + M)\psi$$

because $\gamma^0 D_0 = (-i\gamma^4)(iD_4)$, $\bar{\psi}$ denotes $\psi^\dagger i\gamma^0 = \psi^\dagger\gamma^4$ throughout. For the parity-odd term we have

$$F_{\mu\nu}\tilde{F}^{\mu\nu} \to \pm iF^E_{\mu\nu}\tilde{F}^E_{\mu\nu}$$

because the sum $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ contains exactly one term with an index 0 which gets multiplied with i. The sign depends on the convention adopted for ϵ_{1234}, it is irrelevant for our purposes.
Put together the analytic continuation of the ground state formula is

$$e^{-V_4 E_0} = \int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left\{\int \left(\frac{-1}{4}\,\mathrm{Tr}\, F^E_{\mu\nu}F^E_{\mu\nu} - \bar{\psi}(D_\mu\gamma_\mu + M)\psi + i\lambda F^E_{\mu\nu}\tilde{F}^E_{\mu\nu}\right) \mathrm{d}^4x_E\right\}.$$

where V_4 denotes the volume of Euclidean space. We can integrate out the fermions according to

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-\int \mathrm{d}^4x_E \bar{\psi}B\psi\right) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(-i\int \mathrm{d}^4x\bar{\psi}B\psi\right) = \det B$$

giving us

$$e^{-V_4 E_0} = \int \mathcal{D}A_\mu \det\left(D_\mu\gamma_\mu + M\right) \exp\left\{\int \left(\frac{-1}{4}\,\mathrm{Tr}\, F^E_{\mu\nu}F^E_{\mu\nu} + i\lambda F^E_{\mu\nu}\tilde{F}^E_{\mu\nu}\right) \mathrm{d}^4x_E\right\}.$$

Because of $\gamma_k^\dagger = \gamma_k$, $\not{D}^\dagger = -\not{D}$, therefore $i\not{D}$ is a self-adjoint operator. Because QCD is a vector-like theory which does not distinguish between left-handed and right-handed particles, the eigenvalues of $i\not{D}$ are paired in the following way:

$$i\not{D}\psi = \alpha\psi \Rightarrow i\not{D}(\gamma_5\psi) = -i\gamma_5\not{D}\psi = -\alpha(\gamma_5\psi)$$

with α a real number. Therefore we have

$$\det(\not{D} + M) = \prod_\alpha (M - i\alpha) = \prod_{\alpha>0}(M + i\alpha)(M - i\alpha)$$

where we have omitted zero eigenvalues of $\not{D}$ which contribute factors of M which we have assumed to be positive definite. Therefore $\det(D_\mu\gamma_\mu + M)$ is a positive function of the gauge field A.
From this we see that since $F^E_{\mu\nu}\tilde{F}^E_{\mu\nu} \geq 0$, a value $\lambda \neq 0$ does only add a phase to the exponential which can only make the integral over the field configurations smaller.

$$e^{-V_4 E_0^{\lambda=0}} > e^{-V_4 E_0^{\lambda\neq 0}}$$

from which we conclude that $F_{\mu\nu}\tilde{F}_{\mu\nu}$ cannot get a vacuum expectation value. Therefore if, as in the axion theory, the coefficient λ is effectively dynamical, QCD will choose a vacuum with $\lambda = 0$.
The same reasoning applies to any parity-odd term, any parity-odd term will involve an odd power of terms which involve exactly one F_{0i} and which do therefore give rise to phases when continued to euclidean space.

References

[VW84a] Cumrun Vafa and Edward Witten. Parity conservation in quantum chromodynamics. *Phys. Rev. Lett.*, 53(6):535–536, Aug 1984.

[VW84b] Cumrun Vafa and Edward Witten. Restrictions on symmetry breaking in vector-like gauge theories. *Nuclear Physics B*, 234(1):173 – 188, 1984.

[Wei95] Steven Weinberg. *The quantum theory of fields*. Cambridge University Press, 1995.