

Exercise 7.1 Topological charge

1. We start by writing the contraction of the field strength tensor with its dual in terms of A :

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c)(\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a - gf^{ade}A_\rho^d A_\sigma^e).$$

We are using the antisymmetry under interchange of any two Lorentz indices:

$$\begin{aligned} \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(2\partial_\mu A_\nu^a - gf^{abc}A_\mu^b A_\nu^c)(2\partial_\rho A_\sigma^a - gf^{ade}A_\rho^d A_\sigma^e) \\ &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[4(\partial_\mu A_\nu^a)(\partial_\rho A_\sigma^a) - 2gf^{ade}(\partial_\mu A_\nu^a)A_\rho^d A_\sigma^e - 2gf^{abc}(\partial_\rho A_\sigma^a)A_\mu^b A_\nu^c \\ &\quad + g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A_\rho^d A_\sigma^e]. \end{aligned}$$

We rename $d \rightarrow b$ and $e \rightarrow c$ in the second term and interchange ρ with μ and σ with ν in the second. The last term is symmetric under interchange of μ and ρ and is therefore vanishing in the contraction. We have:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a = \epsilon^{\mu\nu\rho\sigma}(2(\partial_\mu A_\nu^a)(\partial_\rho A_\sigma^a) - 2gf^{abc}(\partial_\mu A_\nu^a)A_\rho^b A_\sigma^c).$$

We turn to the total derivative:

$$\begin{aligned} \partial_\mu K^\mu &= \epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A_\nu^a F_{\rho\sigma}^a + \frac{g}{3}f^{abc}A_\nu^a A_\rho^b A_\sigma^c\right) \\ &= \epsilon^{\mu\nu\rho\sigma}\left[(\partial_\mu A_\nu^a)(2\partial_\rho A_\sigma^a - gf^{abc}A_\rho^b A_\sigma^c) \right. \\ &\quad \left. + A_\nu^a(2\partial_\mu\partial_\nu A_\sigma^a - gf^{abc}(\partial_\mu A_\rho^b)A_\sigma^c - gf^{abc}A_\rho^b(\partial_\mu A_\sigma^c)) \right. \\ &\quad \left. + gf^{abc}((\partial_\mu A_\nu^a)A_\rho^b A_\sigma^c)\right]. \end{aligned}$$

On the second line, we use the antisymmetry under interchange of μ and ν to get rid of the second derivative, on the third line we have used the fact that the antisymmetry of f^{abc} and $\epsilon^{\mu\nu\rho\sigma}$ compensates. We collect the structure constant terms to have

$$\partial_\mu K^\mu = \epsilon^{\mu\nu\rho\sigma}(2(\partial_\mu A_\nu^a)(\partial_\rho A_\sigma^a) - 2gf^{abc}(\partial_\mu A_\nu^a)A_\rho^b A_\sigma^c).$$

2. We use

$$\partial_\mu(\Lambda\Lambda^{-1}) = 0 = \Lambda\partial_\mu\Lambda^{-1} + (\partial_\mu\Lambda)\Lambda^{-1}$$

to show

$$\begin{aligned} [A_\mu, A_\nu] &= \frac{-1}{g^2}((\partial_\mu\Lambda)\Lambda^{-1}(\partial_\nu\Lambda)\Lambda^{-1} - (\partial_\nu\Lambda)\Lambda^{-1}(\partial_\mu\Lambda)\Lambda^{-1}) \\ &= \frac{-1}{g^2}(-(\partial_\mu\Lambda)\Lambda^{-1}\Lambda(\partial_\nu\Lambda^{-1}) + (\partial_\nu\Lambda)\Lambda^{-1}\Lambda(\partial_\mu\Lambda^{-1})) \\ &= \frac{-1}{g^2}((\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})) \end{aligned}$$

and we insert

$$\begin{aligned}
\partial_\mu A_\nu - \partial_\nu A_\mu &= \frac{i}{g} (\partial_\mu ((\partial_\nu \Lambda) \Lambda^{-1}) - \partial_\nu ((\partial_\mu \Lambda) \Lambda^{-1})) \\
&= \frac{i}{g} ((\partial_\mu \partial_\nu \Lambda) \Lambda^{-1} + (\partial_\nu \Lambda) (\partial_\mu \Lambda^{-1}) - (\partial_\nu \partial_\mu \Lambda) \Lambda^{-1} - (\partial_\mu \Lambda) (\partial_\nu \Lambda^{-1})) \\
&= \frac{i}{g} ((\partial_\nu \Lambda) (\partial_\mu \Lambda^{-1}) - (\partial_\mu \Lambda) (\partial_\nu \Lambda^{-1}))
\end{aligned}$$

giving us

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu] = 0.$$

We insert $F_{\mu\nu} = 0$ and $f^{abc} = -2i \text{Tr} [T^a, T^b] T^c$ to arrive at the form on the exercise sheet.

3. We have

$$\Lambda^{-1} = \frac{\mathbf{x}^2 - d^2}{\mathbf{x}^2 + d^2} \mathbf{1} - i \frac{2d}{\mathbf{x}^2 + d^2} x_k \tau_k$$

and therefore

$$\begin{aligned}
A_i &= \frac{i}{g (\mathbf{x}^2 + d^2)^3} [x_i (-4id(\mathbf{x}^2 + d^2)x_k \tau_k) + \tau_i (2id(\mathbf{x}^4 - d^4)) + i\epsilon_{ikr} x_k \tau_r 4(d^2(\mathbf{x}^2 + d^2))] \\
&= \frac{-2d}{g(\mathbf{x}^2 + d^2)^2} [(\mathbf{x}^2 - d^2)\tau_i - 2(x_j \tau_j)x_i + 2d \epsilon_{ijk} x_j \tau_k].
\end{aligned}$$

4. The calculation of the trace is somewhat tedious, we arrive at

$$n = \int d^3x \frac{4d^3}{\pi^2} \frac{1}{(\mathbf{x}^2 + d^2)^3} = 1.$$

Exercise 7.2 Electric dipole moment of the neutron

The two classic papers calculating the relation between $\bar{\theta}$ and the neutron electric dipole moment are [1] and [2], their results are pretty similar. We follow the former¹.

We start from a Lagrangian with real mass term ($\bar{\theta} := \theta - \arg \det M = \theta$), we transform away the anomaly term using a chiral rotation in the quark fields:

$$\begin{aligned}
\psi_k(x) &\rightarrow e^{i\alpha_k \gamma_5} \psi(x) \\
\theta &\rightarrow \theta + 2 \sum_k \alpha_k \\
\bar{\psi}_k m_k P_R \psi_k + \text{h.c.} &\rightarrow \bar{\psi}_k m_k e^{2i\alpha_k} \psi_k + \text{h.c.} = \bar{\psi}_k \tilde{m}_k P_R \psi_k + \text{h.c.}
\end{aligned}$$

where the sum runs over quark flavours and $P_R = (1 + \gamma_5)/2$. If we want θ to vanish we have therefore $\arg \det \tilde{M} = -\bar{\theta}$ ($M = \text{diag}(m_1, \dots, m_k)$). Assuming $|\bar{\theta}| \ll 1$, we relate

$$\begin{aligned}
-\bar{\theta} &= \arg \det \tilde{M} = \arg \det(M + i\eta) = \arg((m_u + i\eta)(m_d + i\eta)(m_s + i\eta)) \\
&\approx \arg(m_u m_d m_s + i\eta(m_u m_d + m_d m_s + m_s m_u)) \\
&\approx \eta \left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)
\end{aligned}$$

¹Note that the conventions used for $\bar{\theta}$ in [1] are not the same as the ones used in the lecture. In our notation, it uses

$$\bar{\theta} = \frac{1}{n_f} \arg \det \tilde{M}$$

with n_f the number of light quark flavours which leads to the above difference in \mathcal{L}_{CP} .

and therefore we have

$$\begin{aligned}\tilde{M} &\approx M - i\bar{\theta} \left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1} \\ \mathcal{L}_{CP} &\approx -\bar{\theta} \left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1} i\bar{\psi}\gamma_5\psi.\end{aligned}$$

We work in time-independent perturbation theory, expanding to first power in $\bar{\theta}$. Due to the CP -violating term in the Lagrangian, the neutron acquires an admixture of baryons with spin 1/2 odd under parity:

$$|n\rangle = |N\rangle + \sum_{N^*} \frac{1}{E_{N^*} - E_N} |N^*\rangle \left\langle N^* \left| \int d^3x \mathcal{L}_{CP} \right| N \right\rangle.$$

Therefore we have for the electric dipole moment:

$$\begin{aligned}d_n &= \langle n | d | n \rangle = 2 \sum_{N^*} \frac{1}{E_{N^*} - E_N} \Re \left(\langle N | d | N^* \rangle \left\langle N^* \left| \int d^3x \mathcal{L}_{CP} \right| N \right\rangle \right) \\ &= -2\bar{\theta} \left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1} \sum_{N^*} \frac{1}{E_{N^*} - E_N} \Re \left(\langle N | d | N^* \rangle \left\langle N^* \left| \int d^3x i\bar{\psi}\gamma_5\psi \right| N \right\rangle \right).\end{aligned}$$

To estimate the order of magnitude (and to circumvent the calculation of the two relevant matrix elements using the MIT bag model in [1]), we restrict the summation to the lowest-lying resonance $N(1535)$, and we insert the numerical values from [3]

$$\begin{aligned}\frac{1}{\Delta E} &\approx \frac{1}{1535 \text{ MeV} - 940 \text{ MeV}} \\ m_s &\approx 105 \text{ MeV}, \quad \frac{m_s}{m_d} \approx 20, \quad \frac{m_d}{m_u} \approx 2 \\ \langle N | d | N^* \rangle &\approx e \cdot r_P \approx e \cdot 0.9 \cdot 10^{-13} \text{ cm}, \quad \left\langle N^* \left| \int d^3x i\bar{\psi}\gamma_5\psi \right| N \right\rangle \approx 1\end{aligned}$$

where r_P denotes the proton charge radius. This results in $d_n \approx 5 \cdot 10^{-16} |\theta| e \text{ cm}$, from $d_n < 0.3 \cdot 10^{-23} e \text{ cm}$ we have therefore $|\theta| \lesssim 0.6 \cdot 10^{-8}$.

References

- [1] Varouzhan Baluni. cp -nonconserving effects in quantum chromodynamics. *Phys. Rev. D*, 19(7):2227–2230, Apr 1979.
- [2] R.J. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten. Chiral estimate of the electric dipole moment of the neutron in quantum chromodynamics. *Physics Letters B*, 88(1-2):123 – 127, 1979.
- [3] C. Amsler et al. Review of particle physics. *Physics Letters B*, 667(1-5):1 – 6, 2008. Review of Particle Physics.