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Advanced Field Theory
Solution 7

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Exercise 7.1 Topological charge

1. We start by writing the contraction of the field strength tensor with its dual in terms of A:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu\right)\left(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho - gf^{ade}A^d_\rho A^e_\sigma\right).$$

We are using the antisymmetry under interchange of any two Lorentz indices:

$$\begin{aligned}\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(2\partial_\mu A^a_\nu - gf^{abc}A^b_\mu A^c_\nu\right)\left(2\partial_\rho A^a_\sigma - gf^{ade}A^d_\rho A^e_\sigma\right)\\ &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left[4(\partial_\mu A^a_\nu)(\partial_\rho A^a_\sigma) - 2gf^{ade}(\partial_\mu A^a_\nu)A^d_\rho A^e_\sigma - 2gf^{abc}(\partial_\rho A^a_\sigma)A^b_\mu A^c_\nu\right.\\ &\qquad \left.+g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\rho A^e_\sigma\right].\end{aligned}$$

We rename $d \to b$ and $e \to c$ in the second term and interchange ρ with μ and σ with ν in the second. The last term is symmetric under interchange of μ and ρ and is therefore vanishing in the contraction. We have:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\left(2(\partial_\mu A^a_\nu)(\partial_\rho A^a_\sigma) - 2gf^{abc}(\partial_\mu A^a_\nu)A^b_\rho A^c_\sigma\right).$$

We turn to the total derivative:

$$\begin{aligned}\partial_\mu K^\mu &= \epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^a_\nu F^a_{\rho\sigma} + \frac{g}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right)\\ &= \epsilon^{\mu\nu\rho\sigma}\left[(\partial_\mu A^a_\nu)(2\partial_\rho A^a_\sigma - gf^{abc}A^b_\rho A^c_\sigma)\right.\\ &\qquad + A^a_\nu\left(2\partial_\mu\partial_\nu A^a_\sigma - gf^{abc}(\partial_\mu A^b_\rho)A^c_\sigma - gf^{abc}A^b_\rho(\partial_\mu A^c_\sigma)\right)\\ &\qquad \left.+gf^{abc}\left((\partial_\mu A^a_\nu)A^b_\rho A^c_\sigma\right)\right].\end{aligned}$$

On the second line, we use the antisymmetry under interchange of μ and ν to get rid of the second derivative, on the third line we have used the fact that the antisymmetry of f^{abc} and $\epsilon^{\mu\nu\rho\sigma}$ compensates. We collect the structure constant terms to have

$$\partial_\mu K^\mu = \epsilon^{\mu\nu\rho\sigma}\left(2(\partial_\mu A^a_\nu)(\partial_\rho A^a_\sigma) - 2gf^{abc}(\partial_\mu A^a_\nu)A^b_\rho A^c_\sigma\right).$$

2. We use

$$\partial_\mu(\Lambda\Lambda^{-1}) = 0 = \Lambda\partial_\mu\Lambda^{-1} + (\partial_\mu\Lambda)\Lambda^{-1}$$

to show

$$\begin{aligned}[A_\mu, A_\nu] &= \frac{-1}{g^2}\left((\partial_\mu\Lambda)\Lambda^{-1}(\partial_\nu\Lambda)\Lambda^{-1} - (\partial_\nu\Lambda)\Lambda^{-1}(\partial_\mu\Lambda)\Lambda^{-1}\right)\\ &= \frac{-1}{g^2}\left(-(\partial_\mu\Lambda)\Lambda^{-1}\Lambda(\partial_\nu\Lambda^{-1}) + (\partial_\nu\Lambda)\Lambda^{-1}\Lambda(\partial_\nu\Lambda^{-1})\right)\\ &= \frac{-1}{g^2}\left((\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})\right)\end{aligned}$$

and we insert

$$\begin{aligned}\partial_\mu A_\nu - \partial_\nu A_\mu &= \frac{i}{g}\left(\partial_\mu\left((\partial_\nu\Lambda)\Lambda^{-1}\right) - \partial_\nu\left((\partial_\mu\Lambda)\Lambda^{-1}\right)\right)\\ &= \frac{i}{g}\left((\partial_\mu\partial_\nu\Lambda)\Lambda^{-1} + (\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\nu\partial_\mu\Lambda)\Lambda^{-1} - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})\right)\\ &= \frac{i}{g}\left((\partial_\nu\Lambda)(\partial_\mu\Lambda^{-1}) - (\partial_\mu\Lambda)(\partial_\nu\Lambda^{-1})\right)\end{aligned}$$

giving us

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right] = 0.$$

We insert $F_{\mu\nu} = 0$ and $f^{abc} = -2i\,\mathrm{Tr}\left[T^a, T^b\right]T^c$ to arrive at the form on the exercise sheet.

3. We have

$$\Lambda^{-1} = \frac{\mathbf{x}^2 - d^2}{\mathbf{x}^2 + d^2}\mathbf{1} - i\frac{2d}{\mathbf{x}^2 + d^2}x_k\tau_k$$

and therefore

$$\begin{aligned}A_i &= \frac{i}{g}\frac{1}{(\mathbf{x}^2 + d^2)^3}\left[x_i\left(-4id(\mathbf{x}^2 + d^2)x_k\tau_k\right) + \tau_i\left(2id(\mathbf{x}^4 - d^4)\right) + i\epsilon_{ikr}x_k\tau_r 4\left(d^2(\mathbf{x}^2 + d^2)\right)\right]\\ &= \frac{-2\,d}{g(\mathbf{x}^2 + d^2)^2}\left[(\mathbf{x}^2 - d^2)\tau_i - 2(x_j\tau_j)x_i + 2d\,\epsilon_{ijk}x_j\tau_k\right].\end{aligned}$$

4. The calculation of the trace is somewhat tedious, we arrive at

$$n = \int \mathrm{d}^3x\,\frac{4d^3}{\pi^2}\frac{1}{(\mathbf{x}^2 + d^2)^3} = 1.$$

Exercise 7.2 Electric dipole moment of the neutron

The two classic papers calculating the relation between $\bar\theta$ and the neutron electric dipole moment are [1] and [2], their results are pretty similar. We follow the former[1].
We start from a Lagrangian with real mass term ($\bar\theta := \theta - \arg\det M = \theta$), we transform away the anomaly term using a chiral rotation in the quark fields:

$$\begin{aligned}\psi_k(x) &\to e^{i\alpha_k\gamma_5}\psi(x)\\ \theta &\to \theta + 2\sum_k \alpha_k\\ \bar\psi_k m_k P_R\psi_k + \text{h.c.} &\to \bar\psi_k m_k e^{2i\alpha_k}\psi_k + \text{h.c.} = \bar\psi_k\tilde m_k P_R\psi_k + \text{h.c.}.\end{aligned}$$

where the sum runs over quark flavours and $P_R = (1+\gamma_5)/2$. If we want θ to vanish we have therefore $\arg\det\tilde M = -\bar\theta$ ($M = \mathrm{diag}(m_1, \ldots, m_k)$). Assuming $\left|\bar\theta\right| \ll 1$, we relate

$$\begin{aligned}-\bar\theta = \arg\det\tilde M &= \arg\det(M + i\eta) = \arg\left((m_u + i\eta)(m_d + i\eta)(m_s + i\eta)\right)\\ &\approx \arg\left(m_u m_d m_s + i\eta(m_u m_d + m_d m_s + m_s m_u)\right)\\ &\approx \eta\left(\frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s}\right)\end{aligned}$$

[1] Note that the conventions used for $\bar\theta$ in [1] are not the same as the ones used in the lecture. In our notation, it uses

$$\bar\theta = \frac{1}{n_f}\arg\det\tilde M$$

with n_f the number of light quark flavours which leads to the above difference in $\mathcal{L}_{CP}$.

and therefore we have

$$\tilde{M} \approx M - i\bar{\theta}\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)^{-1}$$
$$\mathcal{L}_{CP} \approx -\bar{\theta}\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)^{-1} i\bar{\psi}\gamma_5\psi.$$

We work in time-independent perturbation theory, expanding to first power in $\bar{\theta}$. Due to the CP-violating term in the Lagrangian, the neutron acquires an admixture of baryons with spin 1/2 odd under parity:

$$|n\rangle = |N\rangle + \sum_{N^*} \frac{1}{E_{N^*}-E_N} |N^*\rangle \left\langle N^* \middle| \int \mathrm{d}^3x \mathcal{L}_{CP} \middle| N \right\rangle .$$

Therefore we have for the electric dipole moment:

$$\begin{aligned} d_n = \langle n |d| n\rangle &= 2\sum_{N^*} \frac{1}{E_{N^*}-E_N} \Re\left(\langle N |d| N^*\rangle \left\langle N^* \middle| \int \mathrm{d}^3x \mathcal{L}_{CP} \middle| N \right\rangle \right) \\ &= -2\bar{\theta}\left(\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right)^{-1} \sum_{N^*} \frac{1}{E_{N^*}-E_N} \Re\left(\langle N |d| N^*\rangle \left\langle N^* \middle| \int \mathrm{d}^3x\, i\bar{\psi}\gamma_5\psi \middle| N \right\rangle \right). \end{aligned}$$

To estimate the order of magnitude (and to circumvent the calculation of the two relevant matrix elements using the MIT bag model in [1]), we restrict the summation to the lowest-lying resonance $N(1535)$, and we insert the numerical values from [3]

$$\frac{1}{\Delta E} \approx \frac{1}{1535\,\mathrm{MeV} - 940\,\mathrm{MeV}}$$
$$m_s \approx 105\mathrm{MeV}, \qquad \frac{m_s}{m_d} \approx 20, \qquad \frac{m_d}{m_u} \approx 2$$
$$\langle N |d| N^*\rangle \approx e \cdot r_P \approx e \cdot 0.9 \cdot 10^{-13}\mathrm{cm}, \qquad \left\langle N^* \middle| \int \mathrm{d}^3x\, i\bar{\psi}\gamma_5\psi \middle| N \right\rangle \approx 1$$

where r_P denotes the proton charge radious. This results in $d_n \approx 5 \cdot 10^{-16}\,|\theta|\,e\,\mathrm{cm}$, from $d_n < 0.3 \cdot 10^{-23} e\,\mathrm{cm}$ we have therefore $|\theta| \lessapprox 0.6 \cdot 10^{-8}$.

References

[1] Varouzhan Baluni. *cp*-nonconserving effects in quantum chromodynamics. *Phys. Rev. D*, 19(7):2227–2230, Apr 1979.

[2] R.J. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten. Chiral estimate of the electric dipole moment of the neutron in quantum chromodynamics. *Physics Letters B*, 88(1-2):123 – 127, 1979.

[3] C. Amsler et al. Review of particle physics. *Physics Letters B*, 667(1-5):1 – 6, 2008. Review of Particle Physics.