ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Advanced Field Theory
Solution 6

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C. Anastasiou
T. Gehrmann

Exercise 6.1 Semileptonic Tau Decay: $\tau^+ \to \bar{\nu}_\tau \pi^+$

We cross $k \to -k$, $p \to -p$, adding an overall (-1) because we have crossed a fermion and a prefactor 1/2 because we are now averaging over the spin of the incoming tau to arrive at

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}_{\ell^- \to \pi^- \nu_\ell}|^2 = 4G_F^2 f_\pi^2 \left(2(q\cdot p)(k\cdot p) - p^2(q\cdot k)\right).$$

We determine the scalar products from $k = p + q$, $k^2 = m_\ell^2$, $p^2 = m_\pi^2$ and $q^2 = 0$ as

$$p\cdot q = \frac{1}{2}(m_\ell^2 - m_\pi^2), \quad k\cdot q = \frac{1}{2}(m_\ell^2 - m_\pi^2), \quad k\cdot p = \frac{1}{2}(m_\ell^2 + m_\pi^2) \tag{1}$$

which we insert to have

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = 2G_F^2 f_\pi^2 m_\ell^4 \left(1 - \frac{m_\pi^2}{m_\ell^2}\right).$$

We combine this with the integrated two-particle phase space

$$\Pi_2 = \frac{1}{4\pi}\frac{1}{2}\left(1 - \frac{m_\pi^2}{m_\ell^2}\right)$$

to arrive at

$$\Gamma = \frac{1}{2m_\ell}\left(\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2\right)\Pi_2 = \frac{1}{8\pi} G_F^2 f_\pi^2 m_\ell^3 \left(1 - \frac{m_\pi^2}{m_\ell^2}\right)^2.$$

Exercise 6.2 Semileptonic Tau Decay: $\tau^+ \to \bar{\nu}_\tau \rho^+$

In analogy to exercise sheet 4, the matrix element is the contraction of the hadronic current with the leptonic current

$$\mathcal{M} = 2G_F g_\rho \bar{v}(k)\gamma^\mu P_L v(q)\epsilon_{\mu,\lambda}(p) = 2G_F g_\rho \bar{v}(k)\not{\epsilon}_\lambda(p) P_L v(q).$$

We average over incoming spins and sum over outgoing spins and polarizations:

$$\frac{1}{2}\sum_{\text{spins, polarisations}} |\mathcal{M}|^2 = \left\langle |\mathcal{M}|^2 \right\rangle = 2G_F^2 g_\rho^2 \,\mathrm{Tr}\left((\not{k} - m_\ell)\gamma^\mu P_L \not{q}\gamma^\nu \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_\rho^2}\right)\right)$$

where we have inserted the polarisation sum for the ρ polarisation vectors already. We expand this into

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2G_F^2 g_\rho^2 \left(-\,\mathrm{Tr}\left((\not{k} - m_\ell)\gamma^\mu P_L \not{q}\gamma_\mu\right) + \frac{1}{m_\rho^2}\,\mathrm{Tr}\left((\not{k} - m_\ell)\not{p} P_L \not{q}\not{p}\right)\right).$$

We observe that the mass vanishes from the trace because the terms involving it have an odd number of Dirac matrices and that we can insert $P_L = 1/2$ inside the trace because the traces involving γ^5 vanish because there are no four linearly independent momenta in the process. We have therefore

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2G_F^2 g_\rho^2 \left(-\frac{1}{2}\,\mathrm{Tr}\left(\not{k}\gamma^\mu \not{q}\gamma_\mu\right) + \frac{1}{2m_\rho^2}\,\mathrm{Tr}\left(\not{k}\not{p}\not{q}\not{p}\right)\right).$$

Now we use the contraction identity $\gamma_\mu \gamma_\alpha \gamma^\mu = -2\gamma_\alpha$ in the first trace and $\not{p}\not{q} = -\not{q}\not{p} + 2(p\cdot q)$ in the second one to reduce the traces to

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2G_F^2 g_\rho^2 \left(\mathrm{Tr}\left(\not{k}\not{q}\right) - \frac{p^2}{2m_\rho^2} \mathrm{Tr}\left(\not{k}\not{q}\right) + \frac{p\cdot q}{m_\rho^2} \mathrm{Tr}\left(\not{k}\not{p}\right) \right).$$

Now we insert $\mathrm{Tr}(\gamma^\alpha \gamma^\beta) = 4g^{\alpha\beta}$ and $p^2 = m_\rho^2$ to have

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2G_F^2 g_\rho^2 \left(2(k\cdot q) + \frac{4(p\cdot q)(k\cdot p)}{m_\rho^2} \right).$$

We insert equation (1) and expand to arrive at

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2G_F^2 g_\rho^2 \left(m_\ell^2 - 2m_\rho^2 + \frac{m_\ell^4}{m_\rho^2} \right) = 2G_F^2 g_\rho^2 \frac{m_\ell^4}{m_\rho^2} \left(1 - \frac{m_\rho^2}{m_\ell^2}\right) \left(1 + 2\frac{m_\rho^2}{m_\ell^2}\right).$$

From this we get the partial width

$$\Gamma = \frac{1}{2m_\ell} \frac{1}{8\pi} \left(1 - \frac{m_\rho^2}{m_\ell^2}\right) \left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{8\pi} G_F^2 g_\rho^2 \frac{m_\ell^3}{m_\rho^2} \left(1 - \frac{m_\rho^2}{m_\ell^2}\right)^2 \left(1 + 2\frac{m_\rho^2}{m_\ell^2}\right).$$