Advanced Field Theory
Solution 4

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Exercise 4.1 Weak Decay of the Pion

1. For convenience, we define

$$\tau^+ = \frac{1}{\sqrt{2}}\left(\tau^1 + i\tau^2\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned}
\bar{u}_L\gamma^\mu d_L &= \bar{u}\gamma^\mu P_L d \\
&= \bar{u}\gamma^\mu \frac{1}{2}(1-\gamma^5)d \\
&= \frac{1}{2}(\bar{u}\ \bar{d})\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\left(\gamma^\mu - \gamma^\mu\gamma^5\right)\right)\begin{pmatrix} u \\ d \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\left(J^{\mu\,+} - J^{\mu 5\,+}\right).
\end{aligned}$$

2. We recall

$$\begin{aligned}
\left\langle 0 \left| J^{\mu 5\,-} \right| \pi^+(p)\right\rangle &= \left\langle 0 \left| \frac{1}{\sqrt{2}}\left(J^{\mu 5\,1} - iJ^{\mu 5\,2}\right)\right| \frac{1}{\sqrt{2}}\left(\pi^1 + i\pi^2\right)\right\rangle \\
&= -ip^\mu f_\pi e^{-ipx}.
\end{aligned}$$

Due to $(\gamma^5)^\dagger = \gamma^5$, we have

$$\left(J^\mu\right)^\dagger = J^\mu \quad \left(J^{\mu 5}\right)^\dagger = -J^{\mu 5}.$$

and therefore

$$\begin{aligned}
\left(J^{\mu\,+}\right)^\dagger &= \left(J^{\mu\,1} + iJ^{\mu\,2}\right)^\dagger = J^{\mu\,-} \\
\left(J^{\mu 5\,+}\right)^\dagger &= \left(J^{\mu 5\,1} + iJ^{\mu 5\,2}\right)^\dagger = -J^{\mu 5\,-}
\end{aligned}$$

We spell out $\mathcal{L}$:

$$\mathcal{L} = 2G_F\left((\bar{\ell}\gamma^\mu P_L\nu)(J^+_\mu - J^{5\,+}_\mu) + (\bar{\nu}\gamma^\mu P_L\ell)(J^-_\mu + J^{5\,-}_\mu)\right)$$

and we write out $\langle \text{out}\,|L|\,\text{in}\rangle$ explicitly as

$$2\,G_F\left\langle \ell^+(k),\nu(q)\left|\int \mathrm{d}^4x\left((\bar{\ell}\gamma^\mu P_L\nu)(J^+_\mu - J^{5\,+}_\mu) + (\bar{\nu}\gamma^\mu P_L\ell)(J^-_\mu + J^{5\,-}_\mu)\right)\right|\pi^+(p)\right\rangle.$$

The first term vanishes because the fermions are on the wrong side, we execute $\int \mathrm{d}^4x\, e^{i(k+q-p)x}$ to get the delta function:

$$\left\langle \ell^+(k),\nu(q)\left|\int \mathrm{d}^4x\,\mathcal{L}\right|\pi^+(p)\right\rangle = 2\,G_F\bar{u}(q)\gamma^\mu P_L v(k)(-ip_\mu f_\pi)(2\pi)^4\delta^{(4)}(p-q-k).$$

We split off the momentum conserving delta function, arriving at

$$i\mathcal{M} = 2\,G_F\bar{u}(q)\not{p}P_L v(k)\cdot f_\pi.$$

3. Now we want to calculate the decay width of $\pi^+ \to e^+\nu$. We sum over final-state spins to have

$$\sum_{\text{spins}} |\mathcal{M}|^2 = G_F^2 f_\pi^2 \, \text{Tr}\left(\not{q}\not{p}(1-\gamma^5)(\not{k}-m)(1+\gamma^5)\not{p}\right)$$

inserting $(1-\gamma^5)m(1+\gamma^5) = m(1-\gamma^5\gamma^5) = 0$ and $(1-\gamma^5)\not{k}(1+\gamma^5) = (1-\gamma^5)^2\not{k} = 2(1-\gamma^5)\not{k}$

$$= 2\, G_F^2 f_\pi^2 \, \text{Tr}\left(\not{q}\not{p}(1-\gamma^5)\not{k}\not{p}\right)$$

since we have only three linearly independent momenta, the γ^5 trace has to vanish

$$= 8\, G_F^2 f_\pi^2 \left(2(q\cdot p)(k\cdot p) - p^2(q\cdot k)\right).$$

Since we are dealing with a $1 \to 2$ process, all the invariants can be expressed in terms of masses. Starting from $p = q + k$, $p^2 = m_\pi^2$, $k^2 = m_\ell^2$, $q^2 = 0$ we have

$$(q\cdot p) = \frac{1}{2}(m_\pi^2 - m_\ell^2), \quad (k\cdot p) = \frac{1}{2}(m_\pi^2 + m_\ell^2), \quad (q\cdot k) = \frac{1}{2}(m_\pi^2 - m_\ell^2).$$

We insert this:

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}|^2 &= 8\, G_F^2 f_\pi^2 \left(\frac{1}{2}(m_\pi^2 - m_\ell^2)(m_\pi^2 + m_\ell^2) - \frac{1}{2}m_\pi^2(m_\pi^2 - m_\ell^2)\right)\\ &= 4\, G_F^2 f_\pi^2 m_\pi^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right).\end{aligned}$$

Since we have no angular dependence at all, we only need the integrated two-particle phase space

$$\int \mathrm{d}\Pi_2 = \frac{1}{4\pi}\frac{|\mathbf{p}|}{E_{\text{cm}}} = \frac{1}{4\pi}\frac{1}{2}\left(1 - \frac{m_\ell^2}{m_\pi^2}\right)$$

which we insert into the expression for the total decay width

$$\begin{aligned}\Gamma &= \frac{1}{2m_\pi}|\mathcal{M}|^2\,\Pi_2\\ &= \frac{1}{4\pi} G_F^2 f_\pi^2 m_\pi m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2.\end{aligned}$$

From this we can see the suppression of the decay into electrons compared to the decay into muons and, using the values given on the exercise sheet, we can determine $f_\pi \approx 90$ MeV.