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Advanced Field Theory
Solution 2

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Exercise 2.1 Axial Anomaly in Massive QED

As a preliminary, we recall the equations of motion for ψ and $\bar{\psi}$ following from $\mathcal{L}$:

$$\left(i(\partial\!\!\!/ + ieA\!\!\!/) - m\right)\psi = 0, \quad \bar{\psi}\left(i(\overleftarrow{\partial\!\!\!/} - ieA\!\!\!/) + m\right) = 0. \tag{1}$$

where $\overleftarrow{\partial\!\!\!/}$ denotes that the derivative acts on the function to the left of it.
We do now compute the derivatives we will encounter in $\partial_\mu J^{\mu 5}$. By the equations of motion (1) we have

$$\bar{\psi}\left(x+\frac{\delta}{2}\right)\overleftarrow{\partial\!\!\!/} = \bar{\psi}\left(x+\frac{\delta}{2}\right)(+i)\left(eA\!\!\!/\left(x+\frac{\delta}{2}\right)+m\right)$$
$$\partial\!\!\!/\psi\left(x-\frac{\delta}{2}\right) = (-i)\left(eA\!\!\!/\left(x-\frac{\delta}{2}\right)+m\right)\psi\left(x-\frac{\delta}{2}\right)$$

further we can expand the Wilson line for small δ:

$$\begin{aligned}\partial\!\!\!/\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2} dz A(z)\right] &= \partial\!\!\!/\left(1 - ie\int_{x-\delta/2}^{x+\delta/2} dz A(z) + \mathcal{O}(\delta^2)\right) \\ &\approx (-ie)\partial\!\!\!/\delta^\nu A_\nu(x).\end{aligned}$$

Now we write down the divergence of the axial current:

$$\begin{aligned}\partial_\mu\left[\operatorname*{symm\,lim}_{\delta\to 0}\left(\bar{\psi}\left(x+\frac{\delta}{2}\right)\gamma^\mu\gamma^5\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2} dz A(z)\right]\psi\left(x-\frac{\delta}{2}\right)\right)\right] &= \operatorname*{symm\,lim}_{\delta\to 0} \\ \bar{\psi}\left(x+\frac{\delta}{2}\right)(+i)\left(eA\!\!\!/\left(x+\frac{\delta}{2}\right)+m\right)\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2} dz A(z)\right]\gamma^5\psi\left(x-\frac{\delta}{2}\right) \\ +\bar{\psi}\left(x+\frac{\delta}{2}\right)\left(-ie\delta^\nu\partial\!\!\!/A_\nu(x)\right)\gamma^5\psi\left(x-\frac{\delta}{2}\right) \\ +\bar{\psi}\left(x+\frac{\delta}{2}\right)(-i)\left(eA\!\!\!/\left(x-\frac{\delta}{2}\right)-m\right)\exp\left[-ie\int_{x-\delta/2}^{x+\delta/2} dz A(z)\right]\gamma^5\psi\left(x-\frac{\delta}{2}\right)\end{aligned}$$

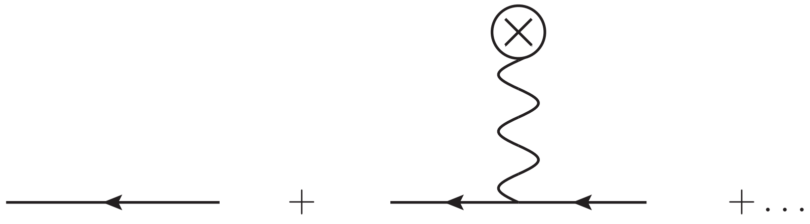
where we have inserted the equations of motion for ψ and $\bar{\psi}$ (the mass from the $\partial\!\!\!/\psi$ term gets a minus sign because we had to anticommute γ^5) and the expansion up to the first order in δ for the derivative of the Wilson line. Since the Wilson line is 1 to first order in δ, we will be able to omit it in the following. We organise the expression as follows:

$$\operatorname*{symm\,lim}_{\delta\to 0}\left[\bar{\psi}\left(x+\frac{\delta}{2}\right)\left((2im) + ie\left(A\!\!\!/\left(x+\frac{\delta}{2}\right) - A\!\!\!/\left(x-\frac{\delta}{2}\right)\right) - ie\delta^\nu\partial\!\!\!/A_\nu(x)\right)\gamma^5\psi\left(x-\frac{\delta}{2}\right)\right].$$

Now we approximate $A\!\!\!/(x+\delta/2) - A\!\!\!/(x-\delta/2) \approx \delta^\nu\partial_\nu A\!\!\!/(x)$ to arrive at

$$\operatorname*{symm\,lim}_{\delta\to 0}\left[\bar{\psi}\left(x+\frac{\delta}{2}\right)\left(2im - ie\gamma^\mu\delta^\nu\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)\right)\gamma^5\psi\left(x-\frac{\delta}{2}\right)\right].$$

Figure 1: The expansion of the fermion propagator in the presence of an external field



We can see that in addition to the term proportionate to the mass, we have a term depending on the vector potential A as well. To evaluate this contribution, we recall that we are actually calculating a vacuum expectation value of the form

$$\langle 0|T\left(\bar{\psi}(y)\Gamma\psi(z)\right)|0\rangle = \mathrm{Tr}\left(\Gamma\langle 0|T\left(\psi(z)\bar{\psi}(y)\right)|0\rangle\right)$$

where Γ denotes an arbitrary product of Dirac matrices. Therefore we do now evaluate the propagator in the presence of the external field A up to the first order in the coupling. The first diagram in figure 1 is the usual Feynman propagator of the noninteracting theory

$$D_F(-\delta) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2}e^{-ip(-\delta)}.$$

To evaluate the limit of the trace for $\delta \to 0$ we can expand the integrand for large values of p:

$$\begin{aligned} D_F(-\delta) &\approx \int \frac{d^4p}{(2\pi)^4}\frac{i\not{p}}{p^2}e^{-ip(-\delta)} \\ &= \not{\partial}\left(\frac{i}{(2\pi)^2}\frac{1}{\delta^2}\right) \\ &= \frac{(-i)}{2\pi^2}\frac{\not{\delta}}{\delta^4}. \end{aligned}$$

Evaluating the trace yields $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$. Evaluating the second term in the expansion (omitting masses from the start in anticipation of the limit $\delta \to 0$) gives

$$\begin{aligned} &\int \mathrm{d}^4b D_F\left(\left(x-\frac{\delta}{2}\right)-b\right)\left(-ie\not{A}(b)\right)D_F\left(b-\left(x+\frac{\delta}{2}\right)\right) \\ &\qquad = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{d}^4b\frac{i\not{p}}{p^2}e^{-ip(x-\delta/2)-b}\left(-ie\not{A}(b)\right)\frac{i\not{k}}{k^2}e^{-ik(b-x-\delta/2)} \end{aligned}$$

executing the Fourier transform on A and shifting $p \to p+k$

$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(\not{p}+\not{k})}{(p+k)^2}\left(-ie\not{A}(p)\right)\frac{i\not{k}}{k^2}e^{-i(x+\delta/2)}e^{ik\delta}.$$

Next we insert this into the trace:

$$\begin{aligned} &\mathrm{Tr}\left(\gamma^\mu\gamma^5\psi\left(x-\frac{\delta}{2}\right)\bar{\psi}\left(x+\frac{\delta}{2}\right)\right) \\ &\quad = \mathrm{Tr}\left(\int \frac{\mathrm{d}^4p}{(2\pi)^4)}\frac{\mathrm{d}^4k}{(2\pi)^4}\gamma^\mu\gamma^5\frac{i(\not{p}+\not{k})}{(p+k)^2}\left(-ie\not{A}(p)\right)\frac{i\not{k}}{k^2}e^{-ip(x+\delta/2)}e^{ik\delta}\right) \\ &\quad = e\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma^5\gamma^\alpha\gamma^\beta\gamma^\rho\right)p_\alpha A_\beta k_\rho\frac{i}{(p+k)^2k^2}e^{-ip(x+\delta/2)}e^{ik\delta}. \end{aligned}$$

Where we have replaced $p+k \to p$ inside the trace because the trace will vanish if a momentum shows up twice. We will approximate $(k+p)^2 \approx k^2$ because we are interested in the limit of small δ, separating the two momentum integrals. We have

$$\begin{aligned}
\epsilon^{\mu\alpha\beta\gamma} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} p_\alpha A_\beta(p) e^{-ip(x+\delta/2)} &= i\epsilon^{\mu\alpha\beta\gamma} \partial_\alpha A_\beta \left(x + \frac{\delta}{2}\right) \\
&\approx i\epsilon^{\mu\alpha\beta\gamma} \partial_\alpha A_\beta\left(x\right) \\
&= \frac{i}{2} \epsilon^{\mu\alpha\beta\gamma} F_{\alpha\beta}(x)
\end{aligned}$$

and

$$\begin{aligned}
\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\gamma}{k^4} e^{ip\delta} &= \left(i \frac{-\partial}{\partial \delta_\gamma}\right) \left(\frac{-1}{16\pi^2} \log \frac{1}{\delta^2}\right) \\
&= \frac{-i}{8\pi^2} \frac{\delta_\gamma}{\delta^2}.
\end{aligned}$$

We insert both of these to arrive at

$$\begin{aligned}
&\operatorname{Tr}\left(\gamma^\mu \gamma^5 \psi \left(x - \frac{\delta}{2}\right) \bar{\psi} \left(x + \frac{\delta}{2}\right)\right) \\
&\approx e\,(-4)\epsilon^{\mu\alpha\beta\rho} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} p_\alpha A_\beta(p) e^{-ip(x+\delta/2)} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\rho}{k^4} e^{ik\delta} \\
&= e\ 2\epsilon^{\alpha\beta\mu\rho} F_{\alpha\beta}(x) \left(\frac{-i}{8\pi^2} \frac{\delta_\rho}{\delta^2}\right).
\end{aligned}$$

We can insert this result into our result for the mass-independent part of the divergence of the axial current:

$$\begin{aligned}
&\operatorname{symm} \lim_{\delta\to 0} \left\langle \bar{\psi}\left(x + \frac{\delta}{2}\right) \left(-ie\delta^\nu \left(F_{\mu\nu}(x)\right) \gamma^\mu \gamma^5 \bar{\psi}\left(x - \frac{\delta}{2}\right)\right\rangle \right. \\
&= \operatorname{symm} \lim_{\delta\to 0} \left(-ie^2 \delta^\nu F_{\mu\nu} 2\epsilon^{\alpha\beta\mu\rho} F_{\alpha\beta} \frac{-i}{8\pi^2} \frac{\delta_\rho}{\delta^2}\right)
\end{aligned}$$

inserting $\frac{\delta_\mu \delta_\nu}{\delta^2} \to \frac{g_{\mu\nu}}{4}$

$$= \frac{-e^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} F_{\alpha\beta}.$$

Exercise 2.2 Fermion Number Nonconservation

1. Let us restate some electrodynamics:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix},$$

furthermore we need to recall $A_\mu = (\phi, -\mathbf{A})$, $\mathbf{E} = -\partial_t \mathbf{A} - \nabla\phi$, $\mathbf{B} = \operatorname{rot} \mathbf{A}$. According to our sign convention $\epsilon^{0123} = 1$ we have therefore

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 8\, \mathbf{E} \cdot \mathbf{B}.$$

Decomposing the axial vector current according to $J^{\mu 5} = \bar\psi\gamma^\mu(P_R - P_L)\psi = \bar\psi_R\gamma^\mu\psi_R - \bar\psi_L\gamma^\mu\psi_L$ we can write (assuming $\mathbf{J}$ vanishes sufficiently fast for $|\mathbf{x}| \to 0$ to have $\int \mathrm{d}^3x\, \partial_i \mathbf{J}^i = 0$)

$$\int \mathrm{d}^4x \partial_\mu J^{\mu 5} = \int \mathrm{d}^4x\, (\partial_\mu J^\mu_R - \partial_\mu J^\mu_L) = N_R|_{t=-\infty}^{t=+\infty} - N_L|_{t=-\infty}^{t=+\infty} = -\frac{e^2}{2\pi^2}\int \mathrm{d}^4x\, \mathbf{E}\cdot\mathbf{B}.$$

2. Since $\mathcal{L}$ does not depend on $\partial_0\bar\psi$ we have

$$\begin{aligned}\mathcal{H} &= \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\partial^0\psi - \mathcal{L}\\ &= i\bar\psi\gamma_0\partial^0\psi - i\bar\psi(\partial\!\!\!/ + ieA\!\!\!/)\psi\\ &= i\bar\psi(\gamma^i\partial_i + ie\gamma^i A_i)\psi + e\bar\psi\gamma^0 A_0\psi\\ &= i\bar\psi(\gamma^i\partial_i - ie\gamma^i\mathbf{A}_i)\psi + e\bar\psi\gamma^0 A_0\psi.\end{aligned}$$

We can write this in terms of ψ_R and ψ_L:

$$\begin{aligned}\bar\psi\gamma^i\psi &= (\psi_R^\dagger\psi_L^\dagger)\begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix}\\ &= \psi_R^\dagger(-\sigma^i)\psi_R + \psi_L^\dagger\sigma^i\psi_L\end{aligned}$$

to arrive at the form on the exercise sheet for $A_0 = 0$.

3. We have $\mathbf{A} = (0, Bx_1, A)$ and we want to solve the Eigenvalue problem $(-i\sigma^i(\partial_i - ie\mathbf{A}_i) - E)\psi_R = 0$. Using the Ansatz on the exercise sheet, we have $\partial_2\psi_R = ik_2\psi_R$, $\partial_3\psi_R = ik_3\psi_R$ which we insert into the differential equation to have

$$\left(-i\sigma^1\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} + \sigma_2(k_2 - eBx_1)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \sigma^3(k_3 - eA)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} - E\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right) = 0$$

where ϕ_1' denotes the derivative with respect to x_1, or, written out explicitly

$$\begin{pmatrix}(k_3 - eA - E) & -i(k_2 - eBx_1)\\ i(k_2 - eBx_1) & -k_3 + eA - E\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = i\begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix}. \tag{2}$$

We solve the second equation for ϕ_2:

$$\phi_2 = \frac{1}{-k_3 + eA - E}i\left(\phi_1' - (k_2 - eBx_1)\phi_1\right)$$

which we differentiate to have

$$\phi_2' = \frac{1}{-k_3 + eA - E}i\left(\phi_1'' - (k_2 - eBx_1)\phi_1' + eB\phi_1\right).$$

We insert this into the first line of (2) to arrive at

$$\phi_1'' + \left(E^2 - (k_3 - eA)^2 - (k_2 - eBx_1)^2 + eB\right)\phi_1 = 0.$$

We recognize that the differential equation for ϕ_1 is the differential equation for a harmonic oscillator centered at $x_1 = k_2/(eB)$. We shift the x_1 coordinate accordingly and introduce $y_1 = \sqrt{eB}x_1$ to bring the equation into the form

$$(-\partial_{y_1}^2 + y_1^2)\phi_1 = \frac{1}{eB}\left(E^2 - (k_3 - eA)^2 + eB\right)\phi_1.$$

From the condition that the equation should have a square-integrable solution, we have the condition

$$E^2 = (k_3 - eA)^2 + n(2eB) \qquad n \in \mathbb{N}_0. \tag{3}$$

4. If the momenta are quantised according to $k_i = (2\pi n_i)/L$ we can translate the condition that the center of the oscillatory motion be inside the cube of length L:

$$0 < \frac{k_2}{eB} < L \Leftrightarrow 0 < \frac{2\pi n_2}{eBL} < L \Leftrightarrow 0 < n_2 < \frac{L^2 eB}{2\pi}.$$

since the energy in (3) is independent of k_2, each energy level is $L^2eB/(2\pi)$-fold degenerate.

5. We consider a shift of the vector potential $A \to A + (2\pi)/(eL)$, this changes (3) as follows:

$$\begin{aligned} E^2 &= \left(\frac{2\pi n_3}{L} - eA\right)^2 + 2eBn \\ &\to \left(\frac{2\pi}{L}(n_3 - 1) - eA\right)^2 + 2eBn \end{aligned}$$

from which we can see that due to the degeneracy of the energy states $L^2eB/(2\pi)$ states for which (3) had a real solution before do not correspond to a real solution anymore.
We can check this result against the Adler-Bell-Jackiw anomaly. We restrict the time interval to $[0, T]$, from $A(t) = A + (2\pi)/L\, t/T$ we determine

$$\frac{-e^2}{2\pi^2}\int_0^L \mathrm{d}^3\mathbf{x} \int_0^T \mathrm{d}t\, \mathbf{E}\cdot\mathbf{B} = \frac{-e^2}{2\pi^2}\int_0^L \mathrm{d}^3\mathbf{x}\int_0^T \mathrm{d}t\,\left(-\frac{2\pi}{eL}B\frac{1}{T}\right) = \frac{eBL^2}{\pi}.$$