Advanced Field Theory
Solution 1

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Exercise 1.1 Dimensional Regularisation

We consider the integral

$$\int \frac{\mathrm{d}^d p_E}{(2\pi)^d} \frac{(p_E^2)^\alpha}{(p_E^2+\Delta)^n}.$$

We change to spherical coordinates, since the dependence is only on p_E^2 we perform the angular integration:

$$\mathrm{d}^d p_E = \frac{1}{2}(p_E^2)^{d/2-1}\mathrm{d}(p_E^2)\mathrm{d}^{d-1}\Omega_d = \frac{1}{2}(p_E^2)^{d/2-1}\mathrm{d}(p_E^2)\frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

We insert this to arrive at the one-dimensional integral

$$\frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int \mathrm{d}(p_E^2)\frac{(p_E^2)^{\alpha+d/2-1}}{(p_E^2+\Delta)^n}$$

which we transform to the integral representation of the beta function according to

$$x = \frac{p_E^2}{p_E^2+\Delta}, \quad p_E^2 = \Delta\frac{x}{1-x}, \quad p_E^2+\Delta = \Delta\frac{1}{1-x}, \quad \mathrm{d}p_E^2 = \Delta\frac{1}{(1-x)^2}$$

arriving at

$$\frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\Delta^{\alpha+d/2-n}\int_0^1 \mathrm{d}x\left(\frac{x}{1-x}\right)^{\alpha+d/2-1}\frac{1}{(1-x)^{n+2}}$$
$$= \frac{1}{(4\pi)^{d/2}}\Delta^{\alpha+d/2-n}\frac{\Gamma(\alpha+d/2)\Gamma(n-\alpha-d/2)}{\Gamma(n)\Gamma(d/2)}.$$

Using $\Gamma(d/2+1) = d/2\Gamma(d/2)$ this specialises to

$$\int \frac{\mathrm{d}^d p_E}{(2\pi)^d}\frac{1}{(p_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\Delta^{d/2-n}\frac{\Gamma(n-d/2)}{\Gamma(n)}$$
$$\int \frac{\mathrm{d}^d p_E}{(2\pi)^d}\frac{p_E^2}{(p_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\Delta^{1+d/2-n}d/2\frac{\Gamma(n-d/2-1)}{\Gamma(n)}.$$

Exercise 1.2 Feynman Parameters

As our starting point, we verify $n=2$ by explicit integration (we will need it for the induction step as well):

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int \mathrm{d}x\mathrm{d}y\frac{x^{a-1}y^{b-1}}{(xC+yD)^{a+b}}\delta(1-x-y)$$
$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int \mathrm{d}x\frac{x^{a-1}(1-x)^{b-1}}{\left(D\left(1-x\frac{D-C}{D}\right)\right)^{a+b}}$$
$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}D^{-a-b}\ {}_2F_1\left(a+b,a,a+b;\frac{D-C}{D}\right)$$
$$= D^{-a-b}\left(\frac{C}{D}\right)^{-a}$$

where we can use the taylor series of $(1-z)^{-a}$ around 0 and the series representation of the hypergeometric function to verify the last step:

$$\left(\frac{\partial}{\partial z}\right)^k (1-z)^{-a} = \frac{\Gamma(a+k)}{\Gamma(a)}(1-z)^{-(a+k)} \Rightarrow (1-z)^{-a} = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)}\frac{z^k}{k!}$$

$$\begin{aligned} {}_2F_1\left(a+b, a, a+b; z\right) &= \sum_{k=0}^{\infty} \frac{\Gamma(a+b+k)}{\Gamma(a+b)}\frac{\Gamma(a+k)}{\Gamma(k)}\frac{\Gamma(a+b)}{\Gamma(a+b+k)}\frac{1}{k!}z^k \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(k)}\frac{z^k}{k!}. \end{aligned}$$

We turn to the induction step. Using the induction hypothesis, we rewrite (all sums and products are from 1 to n):

$$\frac{1}{\left(\prod D_i^{a_i}\right) D_{n+1}^{a_{n+1}}} = \frac{\Gamma(\sum a_i)}{\prod \Gamma(a_i)} \int_0^1 \left(\prod \mathrm{d}x_i\right) \frac{\delta\left(1-\sum x_i\right) \prod x_i^{a_i-1}}{\left(\sum x_i D_i\right)^{\sum a_i}} \frac{1}{D_{n+1}^{a_{n+1}}}.$$

Next, we use the identity for $n=2$ to combine all the denominators

$$\begin{aligned} = &\frac{\Gamma(\sum a_i)}{\prod \Gamma(a_i)} \int_0^1 \left(\prod \mathrm{d}x_i\right) \delta\left(1-\sum x_i\right) \prod x_i^{a_i-1} \\ &\frac{\Gamma\left(\sum a_i + a_{n+1}\right)}{\Gamma\left(\sum a_i\right)\Gamma(a_{n+1})} \int_0^1 \mathrm{d}q\mathrm{d}z \delta\left(1-q-z\right) \frac{q^{(\sum a_i - 1)} z^{a_{n+1}-1}}{\left(q\left(\sum x_i D_i\right) + z D_{n+1}\right)^{\sum a_i + a_{n+1}}} \end{aligned}$$

then we rescale the x_i, $y_i = q x_i$ to have the denominator back in the original form

$$\begin{aligned} = &\frac{\Gamma\left(\sum a_i + a_{n+1}\right)}{\left(\prod \Gamma(a_i)\right)\Gamma(a_{n+1})} \int_0^1 \mathrm{d}q\mathrm{d}z \int_0^q \left(\prod \mathrm{d}y_i\right) q^{-n} \delta\left(1 - \frac{1}{q}\sum y_i\right) \\ &\left(\prod y_i^{a_i-1}\right) q^{(-\sum a_i + n)} \delta(1-q-z) \frac{q^{(\sum a_i - 1)} z^{a_{n+1}-1}}{\left(\sum y_i D_i + z D_{n+1}\right)^{(\sum a_i + a_{n+1})}} \end{aligned}$$

we replace the first delta function $\delta\left(1-\frac{1}{q}\sum y_i\right) = q\ \delta\left(q - \sum x_i\right)$ to get rid of all the factors of q

$$\begin{aligned} = &\frac{\Gamma\left(\sum a_i + a_{n+1}\right)}{\left(\prod \Gamma(a_i)\right)\Gamma(a_{n+1})} \int_0^1 \mathrm{d}q\mathrm{d}z \int_0^q \left(\prod \mathrm{d}y_i\right) \delta\left(q - \sum y_i\right) \\ &\left(\prod y_i^{a_i-1}\right) \delta(1-q-z) \frac{z^{a_{n+1}-1}}{\left(\sum y_i D_i + z D_{n+1}\right)^{(\sum a_i + a_{n+1})}} \end{aligned}$$

finally we perform the q-integral, because of the remaining delta function values of the y_i above $1-z$ do not contribute, so we extend the integrals to go from 0 to 1:

$$\begin{aligned} = &\frac{\Gamma\left(\sum a_i + a_{n+1}\right)}{\left(\prod \Gamma(a_i)\right)\Gamma(a_{n+1})} \int_0^1 \mathrm{d}z \left(\prod \mathrm{d}y_i\right) \delta\left(1 - z - \sum y_i\right) \\ &\frac{\left(\prod y_i^{a_i-1}\right) z^{a_{n+1}-1}}{\left(\sum y_i D_i + z D_{n+1}\right)^{(\sum a_i + a_{n+1})}}. \end{aligned}$$

Exercise 1.3 Generalisation of γ_5 in d Dimensions

The dimension d shows up in the contraction identities:

$$\begin{aligned}
\gamma^\alpha\gamma_\alpha &= \frac{1}{2}\{\gamma^\alpha,\gamma_\alpha\} = g^\alpha_\alpha = d \\
\gamma^\alpha\gamma_\mu\gamma_\alpha &= \gamma^\alpha\left(-\gamma_\alpha\gamma_\mu + 2g_{\alpha\mu}\right) = -(d-2)\gamma_\mu \\
\gamma^\alpha\gamma_\mu\gamma_\nu\gamma_\alpha &= \gamma^\alpha\gamma_\mu\left(-\gamma_\alpha\gamma_\nu + 2g_{\nu\alpha}\right) = (d-2)\gamma_\mu\gamma_\nu + 2\gamma_\nu\gamma_\mu = (d-2)\gamma_\mu\gamma_\nu - 2\gamma_\mu\gamma_\nu + 4g_{\mu\nu} \\
&= (d-4)\gamma_\mu\gamma_\nu + 4g_{\mu\nu}.
\end{aligned}$$

$\epsilon_{\mu\nu\rho\sigma}$ vanishes if two of its indices are equal, so we have (assuming $d \neq 2$ to make $\gamma_\alpha\gamma_\mu\gamma^\alpha$ nonvanishing):

$$\operatorname{Tr}\left(\gamma_\alpha\gamma_\mu\gamma^\alpha\gamma_\nu\gamma_5\right) = 0 = -(d-2)\operatorname{Tr}\left(\gamma_\mu\gamma_\nu\gamma_5\right) \Rightarrow \operatorname{Tr}\left(\gamma_\mu\gamma_\nu\gamma_5\right) = 0. \tag{1}$$

Next we assume $d \neq 2$ and $d \neq 4$, we insert a contraction into $\operatorname{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right)$:

$$\begin{aligned}
\operatorname{Tr}\left(\gamma_\alpha\gamma_\mu\gamma_\nu\gamma^\alpha\gamma_\rho\gamma_\sigma\gamma_5\right) &= (d-4)\operatorname{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) + 4g_{\mu\nu}\underbrace{\operatorname{Tr}\left(\gamma_\rho\gamma_\sigma\gamma_5\right)}_{0} \\
&= -\operatorname{Tr}\left(\gamma^\alpha\gamma_\rho\gamma_\sigma\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_5\right) \\
&= -(d-4)\operatorname{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) - 4g_{\rho\sigma}\underbrace{\operatorname{Tr}\left(\gamma_\mu\gamma_\nu\gamma_5\right)}_{0}
\end{aligned}$$

where we have made use of anticommutation of γ_5 and cyclicity of the trace on the second line. Therefore we have:

$$\operatorname{Tr}\left(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\right) = 0 \qquad (d \neq 2, d \neq 4).$$

Exercise 1.4 Vertex Correction in ϕ^3 Theory

We ignore any prefactors, computing

$$I = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-p_1)^2(k-p_2)^2}.$$

First we insert the Feynman parameter formula we derived in exercise Exercise 1.2, rewriting the denominator as

$$\begin{aligned}
\frac{1}{k^2(k-p_1)^2(k-p_2)^2} &= \Gamma(3)\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\frac{\delta(1-x_1-x_2-x_3)}{\left(x_1k^2 + x_2(k-p_1)^2 + x_3(k-p_2)^2\right)^3} \\
&= \Gamma(3)\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\frac{\delta(1-x_1-x_2-x_3)}{\left((x_1+x_2+x_3)k^2 - 2x_2kp_1 + 2x_3kp_2\right)^3} \\
&= \Gamma(3)\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\frac{\delta(1-x_1-x_2-x_3)}{\left((k-p_1x_2+p_2x_3)^2 + x_2x_3q^2\right)^3}
\end{aligned}$$

where we have used that the delta function implies $x_1 + x_2 + x_3 = 1$. Now we perform the momentum integral according to exercise Exercise 1.1:

$$\begin{aligned}
\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{\left((k)^2 + x_2x_3q^2\right)^3} &= i\int \frac{\mathrm{d}^dk_E}{(2\pi)^d}\frac{1}{\left(-(k)^2 - x_2x_3(-q^2)\right)^3} \\
&= (-i)\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)}\left(x_2x_3(-q^2)\right)^{\frac{d}{2}-3}
\end{aligned}$$

We can now perform the x-integrals and we arrive at

$$I = \frac{-i}{(4\pi)^{d/2}}\Gamma\left(3-\frac{d}{2}\right)\frac{4}{(d-4)^2}\left(-q^2\right)^{\frac{d}{2}-3}.$$