

Exercise 1.1 Dimensional Regularisation

We consider the integral

$$\int \frac{d^d p_E}{(2\pi)^d} \frac{(p_E^2)^\alpha}{(p_E^2 + \Delta)^n}.$$

We change to spherical coordinates, since the dependence is only on p_E^2 we perform the angular integration:

$$d^d p_E = \frac{1}{2} (p_E^2)^{d/2-1} d(p_E^2) d^{d-1} \Omega_d = \frac{1}{2} (p_E^2)^{d/2-1} d(p_E^2) \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

We insert this to arrive at the one-dimensional integral

$$\frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \int d(p_E^2) \frac{(p_E^2)^{\alpha+d/2-1}}{(p_E^2 + \Delta)^n}$$

which we transform to the integral representation of the beta function according to

$$x = \frac{p_E^2}{p_E^2 + \Delta}, \quad p_E^2 = \Delta \frac{x}{1-x}, \quad p_E^2 + \Delta = \Delta \frac{1}{1-x}, \quad dp_E^2 = \Delta \frac{1}{(1-x)^2}$$

arriving at

$$\begin{aligned} \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \Delta^{\alpha+d/2-n} \int_0^1 dx \left(\frac{x}{1-x} \right)^{\alpha+d/2-1} \frac{1}{(1-x)^{n+2}} \\ = \frac{1}{(4\pi)^{d/2}} \Delta^{\alpha+d/2-n} \frac{\Gamma(\alpha+d/2) \Gamma(n-\alpha-d/2)}{\Gamma(n) \Gamma(d/2)}. \end{aligned}$$

Using $\Gamma(d/2 + 1) = d/2 \Gamma(d/2)$ this specialises to

$$\begin{aligned} \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2 + \Delta)^n} &= \frac{1}{(4\pi)^{d/2}} \Delta^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)} \\ \int \frac{d^d p_E}{(2\pi)^d} \frac{p_E^2}{(p_E^2 + \Delta)^n} &= \frac{1}{(4\pi)^{d/2}} \Delta^{1+d/2-n} d/2 \frac{\Gamma(n-d/2-1)}{\Gamma(n)}. \end{aligned}$$

Exercise 1.2 Feynman Parameters

As our starting point, we verify $n = 2$ by explicit integration (we will need it for the induction step as well):

$$\begin{aligned} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int dx dy \frac{x^{a-1} y^{b-1}}{(xC + yD)^{a+b}} \delta(1-x-y) \\ = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int dx \frac{x^{a-1} (1-x)^{b-1}}{(D(1-x \frac{D-C}{D}))^{a+b}} \\ = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} D^{-a-b} {}_2F_1 \left(a+b, a, a+b; \frac{D-C}{D} \right) \\ = D^{-a-b} \left(\frac{C}{D} \right)^{-a} \end{aligned}$$

where we can use the Taylor series of $(1 - z)^{-a}$ around 0 and the series representation of the hypergeometric function to verify the last step:

$$\begin{aligned} \left(\frac{\partial}{\partial z}\right)^k (1 - z)^{-a} &= \frac{\Gamma(a + k)}{\Gamma(a)} (1 - z)^{-(a+k)} \Rightarrow (1 - z)^{-a} = \sum_{k=0}^{\infty} \frac{\Gamma(a + k)}{\Gamma(a)} \frac{z^k}{k!} \\ {}_2F_1(a + b, a, a + b; z) &= \sum_{k=0}^{\infty} \frac{\Gamma(a + b + k)}{\Gamma(a + b)} \frac{\Gamma(a + k)}{\Gamma(k)} \frac{\Gamma(a + b)}{\Gamma(a + b + k)} \frac{1}{k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(a + k)}{\Gamma(k)} \frac{z^k}{k!}. \end{aligned}$$

We turn to the induction step. Using the induction hypothesis, we rewrite (all sums and products are from 1 to n):

$$\frac{1}{(\prod D_i^{a_i}) D_{n+1}^{a_{n+1}}} = \frac{\Gamma(\sum a_i)}{\prod \Gamma(a_i)} \int_0^1 \left(\prod dx_i\right) \frac{\delta(1 - \sum x_i) \prod x_i^{a_i-1}}{(\sum x_i D_i)^{\sum a_i}} \frac{1}{D_{n+1}^{a_{n+1}}}.$$

Next, we use the identity for $n = 2$ to combine all the denominators

$$\begin{aligned} &= \frac{\Gamma(\sum a_i)}{\prod \Gamma(a_i)} \int_0^1 \left(\prod dx_i\right) \delta(1 - \sum x_i) \prod x_i^{a_i-1} \\ &\quad \frac{\Gamma(\sum a_i + a_{n+1})}{\Gamma(\sum a_i) \Gamma(a_{n+1})} \int_0^1 dq dz \delta(1 - q - z) \frac{q^{(\sum a_i-1)} z^{a_{n+1}-1}}{(q(\sum x_i D_i) + z D_{n+1})^{\sum a_i + a_{n+1}}} \end{aligned}$$

then we rescale the $x_i, y_i = qx_i$ to have the denominator back in the original form

$$\begin{aligned} &= \frac{\Gamma(\sum a_i + a_{n+1})}{(\prod \Gamma(a_i)) \Gamma(a_{n+1})} \int_0^1 dq dz \int_0^q \left(\prod dy_i\right) q^{-n} \delta\left(1 - \frac{1}{q} \sum y_i\right) \\ &\quad \left(\prod y_i^{a_i-1}\right) q^{(-\sum a_i + n)} \delta(1 - q - z) \frac{q^{(\sum a_i-1)} z^{a_{n+1}-1}}{(\sum y_i D_i + z D_{n+1})^{(\sum a_i + a_{n+1})}} \end{aligned}$$

we replace the first delta function $\delta\left(1 - \frac{1}{q} \sum y_i\right) = q \delta(q - \sum x_i)$ to get rid of all the factors of q

$$\begin{aligned} &= \frac{\Gamma(\sum a_i + a_{n+1})}{(\prod \Gamma(a_i)) \Gamma(a_{n+1})} \int_0^1 dq dz \int_0^q \left(\prod dy_i\right) \delta(q - \sum y_i) \\ &\quad \left(\prod y_i^{a_i-1}\right) \delta(1 - q - z) \frac{z^{a_{n+1}-1}}{(\sum y_i D_i + z D_{n+1})^{(\sum a_i + a_{n+1})}} \end{aligned}$$

finally we perform the q -integral, because of the remaining delta function values of the y_i above $1 - z$ do not contribute, so we extend the integrals to go from 0 to 1:

$$\begin{aligned} &= \frac{\Gamma(\sum a_i + a_{n+1})}{(\prod \Gamma(a_i)) \Gamma(a_{n+1})} \int_0^1 dz \left(\prod dy_i\right) \delta(1 - z - \sum y_i) \\ &\quad \frac{(\prod y_i^{a_i-1}) z^{a_{n+1}-1}}{(\sum y_i D_i + z D_{n+1})^{(\sum a_i + a_{n+1})}}. \end{aligned}$$

Exercise 1.3 Generalisation of γ_5 in d Dimensions

The dimension d shows up in the contraction identities:

$$\begin{aligned}\gamma^\alpha \gamma_\alpha &= \frac{1}{2} \{\gamma^\alpha, \gamma_\alpha\} = g_\alpha^\alpha = d \\ \gamma^\alpha \gamma_\mu \gamma_\alpha &= \gamma^\alpha (-\gamma_\alpha \gamma_\mu + 2g_{\alpha\mu}) = -(d-2)\gamma_\mu \\ \gamma^\alpha \gamma_\mu \gamma_\nu \gamma_\alpha &= \gamma^\alpha \gamma_\mu (-\gamma_\alpha \gamma_\nu + 2g_{\nu\alpha}) = (d-2)\gamma_\mu \gamma_\nu + 2\gamma_\nu \gamma_\mu = (d-2)\gamma_\mu \gamma_\nu - 2\gamma_\mu \gamma_\nu + 4g_{\mu\nu} \\ &= (d-4)\gamma_\mu \gamma_\nu + 4g_{\mu\nu}.\end{aligned}$$

$\epsilon_{\mu\nu\rho\sigma}$ vanishes if two of its indices are equal, so we have (assuming $d \neq 2$ to make $\gamma_\alpha \gamma_\mu \gamma^\alpha$ nonvanishing):

$$\text{Tr}(\gamma_\alpha \gamma_\mu \gamma^\alpha \gamma_\nu \gamma_5) = 0 = -(d-2) \text{Tr}(\gamma_\mu \gamma_\nu \gamma_5) \Rightarrow \text{Tr}(\gamma_\mu \gamma_\nu \gamma_5) = 0. \quad (1)$$

Next we assume $d \neq 2$ and $d \neq 4$, we insert a contraction into $\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5)$:

$$\begin{aligned}\text{Tr}(\gamma_\alpha \gamma_\mu \gamma_\nu \gamma^\alpha \gamma_\rho \gamma_\sigma \gamma_5) &= (d-4) \text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5) + 4g_{\mu\nu} \underbrace{\text{Tr}(\gamma_\rho \gamma_\sigma \gamma_5)}_0 \\ &= -\text{Tr}(\gamma^\alpha \gamma_\rho \gamma_\sigma \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_5) \\ &= -(d-4) \text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5) - 4g_{\rho\sigma} \underbrace{\text{Tr}(\gamma_\mu \gamma_\nu \gamma_5)}_0\end{aligned}$$

where we have made use of anticommutation of γ_5 and cyclicity of the trace on the second line. Therefore we have:

$$\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5) = 0 \quad (d \neq 2, d \neq 4).$$

Exercise 1.4 Vertex Correction in ϕ^3 Theory

We ignore any prefactors, computing

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p_1)^2(k-p_2)^2}.$$

First we insert the Feynman parameter formula we derived in exercise Exercise 1.2, rewriting the denominator as

$$\begin{aligned}\frac{1}{k^2(k-p_1)^2(k-p_2)^2} &= \Gamma(3) \int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{(x_1 k^2 + x_2(k-p_1)^2 + x_3(k-p_2)^2)^3} \\ &= \Gamma(3) \int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{((x_1+x_2+x_3)k^2 - 2x_2 k p_1 + 2x_3 k p_2)^3} \\ &= \Gamma(3) \int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{((k-p_1 x_2 + p_2 x_3)^2 + x_2 x_3 q^2)^3}\end{aligned}$$

where we have used that the delta function implies $x_1 + x_2 + x_3 = 1$. Now we perform the momentum integral according to exercise Exercise 1.1:

$$\begin{aligned}\int \frac{d^d k}{(2\pi)^d} \frac{1}{((k-p_1 x_2 + p_2 x_3)^2 + x_2 x_3 q^2)^3} &= i \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(-(k)^2 - x_2 x_3 (-q^2))^3} \\ &= (-i) \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(3 - \frac{d}{2})}{\Gamma(3)} (x_2 x_3 (-q^2))^{\frac{d}{2}-3}\end{aligned}$$

We can now perform the x -integrals and we arrive at

$$I = \frac{-i}{(4\pi)^{d/2}} \Gamma\left(3 - \frac{d}{2}\right) \frac{4}{(d-4)^2} (-q^2)^{\frac{d}{2}-3}.$$