ETH
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Advanced Field Theory
Assignment 13

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Exercise 13.1 The Partition Function in $\lambda\phi^4$ Theory

We consider the thermodynamics of a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \lambda\phi^4,$$

with partition function

$$Z = N\int \mathcal{D}\phi e^S, \qquad S = -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2 + \lambda\phi^4\right]$$

where $\beta = 1/T$, τ is the imaginary time variable and the path integral extends over field configurations with $\phi(0) = \phi(\tau)$. We recall $P = T\partial_V \ln Z$, we fix the limit $P \xrightarrow{\beta\to\infty} 0$.

1. Calculate the β-dependent part of $\ln Z$ for $\lambda = 0$. Evaluate the pressure in the massless case.

2. Calculate the first correction to the β-dependent part of $\ln Z$. We have

$$\ln Z_1 = \frac{-\lambda \int \mathrm{d}\tau \int \mathrm{d}^3x \int \mathcal{D}\phi e^{S_0}\phi^4(\mathbf{x},\tau)}{\int \mathcal{D}\phi e^{S_0}}.$$

   where S_0 is the action of the free field. Insert the Fourier transforms of the fields, the normalisations should be chosen to be

$$\phi(\mathbf{x},\tau) = \left(\frac{\beta}{V}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty}\sum_{\mathbf{p}} e^{i(\mathbf{px}+\omega_n\tau)}\phi_n(\mathbf{p})$$

   with $\omega_n = 2\pi nT$,

$$\int \mathrm{d}^3x e^{i\mathbf{px}} = V\delta(\mathbf{p}), \qquad \int \mathrm{d}\tau e^{i\omega\tau} = \beta\delta(\omega).$$

   You will arrive at

$$\ln Z_1 = -3\lambda\beta V\left[T\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}\right]^2.$$

3. Evaluate the pressure at order λ in the massless case.

The following formulae may provide useful:

$$\int_0^\infty \mathrm{d}\omega \ln\left(1 - e^{-\beta\omega}\right)\omega^2 = \frac{-\pi^4}{45}\beta^{-3}.$$

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+x^2} = \frac{\pi}{a}\coth(\pi a)$$

$$\int_0^\infty \mathrm{d}x e^{-\frac{1}{2}x^2} = \sqrt{\frac{\pi}{2}}a^{-\frac{1}{2}}, \qquad \int_0^\infty \mathrm{d}x e^{-\frac{1}{2}x^2}x^2 = \sqrt{\frac{\pi}{2}}a^{-\frac{3}{2}}, \qquad \int_0^\infty \mathrm{d}x e^{-\frac{1}{2}x^2}x^4 = \sqrt{\frac{\pi}{2}}3a^{-\frac{5}{2}}$$