

Exercise 13.1 The Partition Function in $\lambda\phi^4$ Theory

We consider the thermodynamics of a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4,$$

with partition function

$$Z = N \int \mathcal{D}\phi e^S, \quad S = -\frac{1}{2} \int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial \phi}{\partial \tau} \right)^2 + (\nabla \phi)^2 + m^2 \phi^2 + \lambda \phi^4 \right]$$

where $\beta = 1/T$, τ is the imaginary time variable and the path integral extends over field configurations with $\phi(0) = \phi(\beta)$. We recall $P = T \partial_V \ln Z$, we fix the limit $P \xrightarrow{\beta \rightarrow \infty} 0$.

1. Calculate the β -dependent part of $\ln Z$ for $\lambda = 0$. Evaluate the pressure in the massless case.
2. Calculate the first correction to the β -dependent part of $\ln Z$. We have

$$\ln Z_1 = \frac{-\lambda \int d\tau \int d^3x \int \mathcal{D}\phi e^{S_0} \phi^4(\mathbf{x}, \tau)}{\int \mathcal{D}\phi e^{S_0}}.$$

where S_0 is the action of the free field. Insert the Fourier transforms of the fields, the normalisations should be chosen to be

$$\phi(\mathbf{x}, \tau) = \left(\frac{\beta}{V} \right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} e^{i(\mathbf{p}\mathbf{x} + \omega_n \tau)} \phi_n(\mathbf{p})$$

with $\omega_n = 2\pi nT$,

$$\int d^3x e^{i\mathbf{p}\mathbf{x}} = V \delta(\mathbf{p}), \quad \int d\tau e^{i\omega \tau} = \beta \delta(\omega).$$

You will arrive at

$$\ln Z_1 = -3\lambda\beta V \left[T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2} \right]^2.$$

3. Evaluate the pressure at order λ in the massless case.

The following formulae may provide useful:

$$\int_0^\infty d\omega \ln(1 - e^{-\beta\omega}) \omega^2 = \frac{-\pi^4}{45} \beta^{-3}.$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + x^2} = \frac{\pi}{a} \coth(\pi a)$$

$$\int_0^\infty dx e^{-\frac{1}{2}x^2} = \sqrt{\frac{\pi}{2}} a^{-\frac{1}{2}}, \quad \int_0^\infty dx e^{-\frac{1}{2}x^2} x^2 = \sqrt{\frac{\pi}{2}} a^{-\frac{3}{2}}, \quad \int_0^\infty dx e^{-\frac{1}{2}x^2} x^4 = \sqrt{\frac{\pi}{2}} 3a^{-\frac{5}{2}}$$