Advanced Field Theory
Assignment 11

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C. Anastasiou
T. Gehrmann

Exercise 11.1 Magnetic monopoles

Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{\lambda}{8}\left(\phi^2 - \eta^2\right)^2$$

where ϕ is a scalar field in the three-dimensional representation of $SO(3)$, the covariant derivative is given by

$$D_\mu\phi_a = \partial_\mu\phi_a - g\epsilon_{abc}A^b_\mu\phi_c$$

and the field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon_{abc}A^b_\mu A^c_\nu$$

The magnetic monopole solution can be parametrised by the ansatz

$$\phi_a = \frac{H(\xi)}{\xi}\eta\frac{x_a}{r}$$

$$A^a_i = \frac{\epsilon_{aij}x_j}{gr^2}\left(K(\xi) - 1\right)$$

where $\xi = g\eta r$.

1. Show that the covariant derivative is given in term of this ansatz by
$$D_i\phi_a = \frac{K(\xi)H(\xi)}{gr^4}\left(r^2\delta_{ai} - x_a x_i\right) + \left(\xi H'(\xi) - H(\xi)\right)\frac{x_a x_i}{gr^4}$$
2. Show that the energy of the magnetic monopole is then given by
$$\begin{aligned} E &= \frac{4\pi\eta}{g}\int_0^\infty \mathrm{d}\xi \frac{1}{\xi^2}\left[\frac{1}{2}(\xi H' - H)^2 + H^2K^2 + (\xi K')^2\right. \\ &\quad \left. + \frac{1}{2}(K^2-1)^2 + \frac{\lambda}{8g^2}(H^2 - \xi^2)^2\right] \end{aligned}$$
3. Show that this energy is minimised for
$$\begin{aligned} \xi^2 K'' &= KH^2 + K(K^2 - 1) \\ \xi^2 H'' &= 2K^2H + \frac{\lambda}{2g^2}H(H^2 - \xi^2) \end{aligned}$$

Exercise 11.2 Colour decomposition

In large-N QCD, the amplitude for a n-gluon process is given by

$$\mathcal{M} = \sum_\sigma \mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right) M\left(\sigma(1),\ldots,\sigma(n)\right)$$

where the $M(1,\ldots,n)$ are the partial amplitudes, which contain all the kinematic information (momenta and helicities), and σ is a permutation of the colour indices.

Show diagrammatically that at the leading order in the number of colours, the total amplitude squared is proportional to the sum of the partial amplitudes squared

$$|\mathcal{M}|^2 \sim \sum_\sigma \left|M\left(\sigma(1),\ldots,\sigma(n)\right)\right|^2$$