ETH
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Advanced Field Theory
Assignment 9

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Exercise 9.1 Pure gauge

Show that in the case where the gauge field is given by a pure gauge transformation,

$$A_\mu = \frac{i}{g} \left(\partial_\mu \Lambda\right) \Lambda^{-1}$$

where Λ is a $SU(N)$ matrix, then the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ is identically zero.

Exercise 9.2 Instantons

In 4D Euclidean space ($\mu, \nu = 1, 2, 3, 4$), consider the instanton configuration of a $SU(2)$ gauge field

$$\underline{A}^a_\mu = \frac{2}{g} \frac{\eta^a_{\mu\nu} x_\nu}{|x|^2 + \lambda^2}$$

where λ is an arbitrary distance and η is 't Hooft's mixed colour and space-time tensor defined as

$$\eta^a_{\mu\nu} = \epsilon_{a\mu\nu 4} - \delta_{\mu 4}\delta_{a\nu} + \delta_{\nu 4}\delta_{a\mu}$$

or equivalently

$$\eta^a_{\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \text{if } \mu \neq 4,\ \nu \neq 4 \\ \delta_{a\mu} & \text{if } \mu \neq 4,\ \nu = 4 \\ -\delta_{a\nu} & \text{if } \mu = 4,\ \nu \neq 4 \\ 0 & \text{if } \mu = \nu = 4 \end{cases}$$

Some of the properties of η:

$$\begin{aligned} \eta^a_{\mu\nu} &= -\eta^a_{\nu\mu} \\ \epsilon_{abc}\eta^b_{\mu\nu}\eta^c_{\rho\sigma} &= \delta_{\mu\rho}\eta^a_{\nu\sigma} - \delta_{\mu\sigma}\eta^a_{\nu\rho} - \delta_{\nu\rho}\eta^a_{\mu\sigma} + \delta_{\nu\sigma}\eta^a_{\mu\rho} \\ \eta^a_{\mu\nu}\eta^a_{\mu\nu} &= 12 \end{aligned}$$

1. Show that the field strength tensor is given by

$$\underline{F}^a_{\mu\nu} = \frac{4}{g} \frac{-\lambda^2 \eta^a_{\mu\nu}}{\left(|x|^2 + \lambda^2\right)^2}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu$.

2. Compute the action corresponding to this field configuration:

$$\underline{S} = \frac{1}{4} \int \mathrm{d}^4 x \underline{F}^a_{\mu\nu} \underline{F}^a_{\mu\nu}$$