Advanced Field Theory
Assignment 7

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Exercise 7.1 Topological charge

Consider a $SU(N)$ gauge theory, with the covariant derivative defined as

$$D_\mu = \partial_\mu + igA_\mu, \quad A_\mu = A_\mu^a T^a, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c$$

where the T^a are the generators of $SU(N)$, satisfying $[T^a, T^b] = if^{abc}T^c$.

1. Show that $F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = 1/2\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$ can be written as a total derivative, namely

$$F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = \partial_\mu K^\mu \qquad\qquad K^\mu = \epsilon^{\mu\nu\rho\sigma}\left(A_\nu^a F_{\rho\sigma}^a + \frac{g}{3} f_{abc} A_\nu^a A_\rho^b A_\sigma^c\right)$$

2. The topological charge is defined as

$$n = \frac{g^2}{32\pi^2}\int \mathrm{d}^4x F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = \frac{g^2}{32\pi^2}\int \mathrm{d}^3x\, K_0\Big|_{t=-\infty}^{t=+\infty}$$

Consider an adiabatic transformation

$$\begin{aligned} A_\mu(t=-\infty) &= 0 \\ A_\mu(t=+\infty) &= \frac{i}{g}\left(\partial_\mu \Lambda\right)\Lambda^{-1} \qquad\qquad \Lambda = \frac{\mathbf{x}^2 - d^2}{\mathbf{x}^2 + d^2}\mathbf{1} + i\frac{2d}{\mathbf{x}^2 + d^2} x_k \tau_k \end{aligned}$$

where the τ_i are the generators of a $SU(2)$ subgroup: $\tau_i \tau_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\tau_k$.

Check that $F_{\mu\nu} = 0$, so that the topological charge reads

$$n = \frac{g^3}{96\pi^2}\int \mathrm{d}^3x\, \epsilon^{ijk} f^{abc} A_i^a A_j^b A_k^c = -i\frac{g^3}{24\pi^2}\int \mathrm{d}^3x\, \epsilon^{ijk}\,\mathrm{Tr}\, A_i A_j A_k$$

3. Show that $A_0(t=+\infty) = 0$ and

$$A_i(t=+\infty) = \frac{-2\,d}{g(\mathbf{x}^2 + d^2)^2}\left[(\mathbf{x}^2 - d^2)\tau_i - 2(x_j\tau_j)x_i + 2d\,\epsilon_{ijk}x_j\tau_k\right]$$

4. Compute the topological charge n of this adiabatic transformation. We recall the contraction identities

$$\epsilon^{ijk}\epsilon^{ijm} = 2\delta^{km}, \qquad \epsilon^{ijk}\epsilon^{iml} = \delta^{jm}\delta^{kl} - \delta^{jl}\delta^{km}.$$

Exercise 7.2 Electric dipole moment of the neutron

The electric dipole moment of the neutron, denoted by d_E, is defined as

$$\langle n(p')|J_\mu^{em}|n(p)\rangle\big|_{\mathrm{edm}} = id_E\, \bar{u}(p')\sigma_{\mu\nu}(p'-p)^\nu\gamma^5 u(p)$$

No electric diplole moment is produced at first order in the Standard Model. The main contribution to it comes from the θ parameter appearing in the full Lagrangian of QCD:

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_{\mathrm{QCD}}^{(\theta=0)} + \theta\frac{g^2}{32\pi^2}F_{\mu\nu}^a\tilde{F}^{\mu\nu,a}$$

Estimate the neutron electric dipole moment d_E as a function of θ.