Quantum Mechanics II
Solution Sheet 12

FS 09
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Exercise 12.1 Many-Body Perturbation Theory

a) We first consider the case $N = 1$:

$$
\begin{aligned}
\hat{\mathcal{H}}_{0,\mathcal{F}} \,|\, \mathbf{p} \,\rangle &= \hat{\mathcal{H}}_{0,\mathcal{F}} \, \hat{b}^\dagger(\mathbf{p},t) \,|\, 0 \,\rangle \\
&= \int \mathrm{d}^3k \, \frac{\hbar^2 \, |\mathbf{k}|^2}{2m} \, \hat{b}^\dagger(\mathbf{k},t) \, \hat{b}(\mathbf{k},t) \, \hat{b}^\dagger(\mathbf{p},t) \,|\, 0 \,\rangle) \\
&= \pm \int \mathrm{d}^3k \, \frac{\hbar^2 \, |\mathbf{k}|^2}{2m} \, \hat{b}^\dagger(\mathbf{k},t) \, \hat{b}^\dagger(\mathbf{p},t) \, \underbrace{\hat{b}(\mathbf{k},t) \,|\, 0 \,\rangle}_{=0} \\
&\quad + \int \mathrm{d}^3k \, \frac{\hbar^2 \, |\mathbf{k}|^2}{2m} \, \hat{b}^\dagger(\mathbf{k},t) \, \delta^{(3)}(\mathbf{p}-\mathbf{k}) \,|\, 0 \,\rangle \\
&= \frac{\hbar^2 \, |\mathbf{p}|^2}{2m} \, \hat{b}^\dagger(\mathbf{p},t) \,|\, 0 \,\rangle \\
&= \frac{\hbar^2 \, |\mathbf{p}|^2}{2m} |\, \mathbf{p} \,\rangle
\end{aligned} \tag{1}
$$

Next, we assume the statement to be valid for the case of $N-1$ particles and consider the step from $N-1$ to N, where we denote the $N-1$ particle state $|\, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_N \,\rangle$ by $|\, \mathbf{N-1} \,\rangle$:

$$
\begin{aligned}
\hat{\mathcal{H}}_{0,\mathcal{F}} \,|\, \mathbf{p}, \mathbf{p}_2, \ldots, \mathbf{p}_N \,\rangle &= \hat{\mathcal{H}}_{0,\mathcal{F}} \, \hat{b}^\dagger(\mathbf{p},t) \,|\, \mathbf{N-1} \,\rangle \\
&= \int \mathrm{d}^3k \, \frac{\hbar^2 \, |\mathbf{k}|^2}{2m} \, \hat{b}^\dagger(\mathbf{k},t) \, \hat{b}(\mathbf{k},t) \, \hat{b}^\dagger(\mathbf{p},t) \,|\, \mathbf{N-1} \,\rangle \\
&= \pm \int \mathrm{d}^3k \, \frac{\hbar^2 \, |\mathbf{k}|^2}{2m} \, \hat{b}^\dagger(\mathbf{k},t) \, \hat{b}^\dagger(\mathbf{p},t) \, \hat{b}(\mathbf{k},t) \,|\, \mathbf{N-1} \,\rangle \\
&\quad + \int \mathrm{d}^3k \, \frac{\hbar^2 \, |\mathbf{k}|^2}{2m} \, \hat{b}^\dagger(\mathbf{k},t) \, \delta^{(3)}(\mathbf{p}-\mathbf{k}) \,|\, \mathbf{N-1} \,\rangle \\
&= \hat{b}^\dagger(\mathbf{p},t) \int \mathrm{d}^3k \, \frac{\hbar^2 \, |\mathbf{k}|^2}{2m} \, \hat{b}^\dagger(\mathbf{k},t) \, \hat{b}(\mathbf{k},t) \,|\, \mathbf{N-1} \,\rangle \\
&\quad + \frac{\hbar^2 \, |\mathbf{p}|^2}{2m} \, \hat{b}^\dagger(\mathbf{p},t) \,|\, \mathbf{N-1} \,\rangle \\
&= \hat{b}^\dagger(\mathbf{p},t) \, \hat{\mathcal{H}}_{0,\mathcal{F}} \,|\, \mathbf{N-1} \,\rangle \\
&\quad + \frac{\hbar^2 \, |\mathbf{p}|^2}{2m} \, \hat{b}^\dagger(\mathbf{p},t) \,|\, \mathbf{N-1} \,\rangle
\end{aligned} \tag{2}
$$

Now we can use the induction hypothesis,

$$
\hat{\mathcal{H}}_{0,\mathcal{F}} \,|\, \mathbf{N-1} \,\rangle = \sum_{i=2}^{N} \frac{\hbar^2 \, |\mathbf{p}_i|^2}{2m} \,|\, \mathbf{N-1} \,\rangle \,, \tag{3}
$$

which then leads to

$$
\begin{aligned}
\hat{\mathcal{H}}_{0,\mathcal{F}}\,|\,\mathbf{p},\mathbf{p}_2,\ldots,\mathbf{p}_N\,\rangle &= \hat{b}^\dagger(\mathbf{p},t)\sum_{i=2}^{N}\frac{\hbar^2\,|\mathbf{p}_i|^2}{2m}\,|\,\mathbf{N-1}\,\rangle \\
&\quad+\frac{\hbar^2\,|\mathbf{p}|^2}{2m}\,\hat{b}^\dagger(\mathbf{p},t)\,|\,\mathbf{N-1}\,\rangle \\
&=\left(\frac{\hbar^2\,|\mathbf{p}|^2}{2m}+\sum_{i=2}^{N}\frac{\hbar^2\,|\mathbf{p}_i|^2}{2m}\right)\hat{b}^\dagger(\mathbf{p},t)|\,\mathbf{N-1}\,\rangle \\
&=\left(\frac{\hbar^2\,|\mathbf{p}|^2}{2m}+\sum_{i=2}^{N}\frac{\hbar^2\,|\mathbf{p}_i|^2}{2m}\right)|\,\mathbf{p},\mathbf{p}_2,\ldots,\mathbf{p}_N\,\rangle \qquad (4)
\end{aligned}
$$

Note that in this exercise we explicitly denote operators by a "hat". The momenta $\mathbf{p}$, $\mathbf{p}_2$, etc. are numbers and thus commute with $\hat{b}(\mathbf{p},t)$ and $\hat{b}^\dagger(\mathbf{p},t)$.

b) (i)

$$
\begin{aligned}
\langle\,\mathbf{p}\,|\,\mathbf{k}\,\rangle &= \langle\,0\,|\hat{b}(\mathbf{p},t)\hat{b}^\dagger(\mathbf{k},t)|\,0\,\rangle \\
&= \langle\,0\,|\delta^{(3)}(\mathbf{k}-\mathbf{p})\pm\hat{b}^\dagger(\mathbf{k},t)\hat{b}(\mathbf{p},t)|\,0\,\rangle=\delta^{(3)}(\mathbf{k}-\mathbf{p})
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\langle\,\mathbf{k}_1,\mathbf{k}_2\,|\,\mathbf{p}_1,\mathbf{p}_2\,\rangle &= \langle\,0\,|\hat{b}(\mathbf{k}_2,t)\hat{b}(\mathbf{k}_1,t)\hat{b}^\dagger(\mathbf{p}_1,t)\hat{b}^\dagger(\mathbf{p}_2,t)|\,0\,\rangle \\
&= \langle\,0\,|\hat{b}(\mathbf{k}_2,t)\Big(\delta^{(3)}(\mathbf{p}_1-\mathbf{k}_1)\pm\hat{b}^\dagger(\mathbf{p}_1,t)\hat{b}(\mathbf{k}_1,t)\Big)\hat{b}^\dagger(\mathbf{p}_2,t)|\,0\,\rangle \\
&= \delta^{(3)}(\mathbf{p}_2-\mathbf{k}_2)\delta^{(3)}(\mathbf{p}_1-\mathbf{k}_1) \\
&\quad\pm\delta^{(3)}(\mathbf{p}_2-\mathbf{k}_1)\langle\,0\,|\hat{b}(\mathbf{k}_2,t)\hat{b}^\dagger(\mathbf{p}_1,t)|\,0\,\rangle \\
&= \delta^{(3)}(\mathbf{p}_2-\mathbf{k}_2)\delta^{(3)}(\mathbf{p}_1-\mathbf{k}_1)\pm\delta^{(3)}(\mathbf{p}_2-\mathbf{k}_1)\delta^{(3)}(\mathbf{p}_1-\mathbf{k}_2)
\end{aligned}
$$

where the plus (minus) signs corresponds to bosons (fermions).

(iii) The interaction term $\hat{\mathcal{V}}_F$ has two destruction operators to the right and thus, applying it to a one particle state such as $|\,\mathbf{p}\,\rangle$ necessarily destroys the state. This makes sense since it describes the interaction between two particles and can not describe the interaction of a particle with itself.

(iv) We first examine that the effect of two destruction operators on the state $|\,\mathbf{p}_1,\mathbf{p}_2\,\rangle$ is basically described in (ii) and yields

$$
\begin{aligned}
&\hat{b}(\mathbf{k}_2-\mathbf{q})\hat{b}(\mathbf{k}_1+\mathbf{q})\hat{b}^\dagger(\mathbf{p}_1)\hat{b}^\dagger(\mathbf{p}_2)|\,0\,\rangle \\
=\ &\Big(\delta^{(3)}(\mathbf{k}_1+\mathbf{q}-\mathbf{p}_1)\delta^{(3)}(\mathbf{k}_2-\mathbf{q}-\mathbf{p}_2)\pm\delta^{(3)}(\mathbf{k}_1+\mathbf{q}-\mathbf{p}_2)\delta^{(3)}(\mathbf{k}_2-\mathbf{q}-\mathbf{p}_1)\Big)|\,0\,\rangle
\end{aligned}
$$

Integration over d^3k_1 and d^3k_2 then yields

$$
\begin{aligned}
&\int\frac{d^3q}{(2\pi)^{3/2}}\tilde{V}(\mathbf{q})\Big(\hat{b}^\dagger(\mathbf{p}_1-\mathbf{q})\hat{b}^\dagger(\mathbf{p}_2+\mathbf{q})\pm\hat{b}^\dagger(\mathbf{p}_1+\mathbf{q})\hat{b}^\dagger(\mathbf{p}_2-\mathbf{q})\Big)|\,0\,\rangle \\
=\ &\int\frac{d^3q}{(2\pi)^{3/2}}\tilde{V}(\mathbf{q})\Big(|\,\mathbf{p}_1-\mathbf{q},\mathbf{p}_2+\mathbf{q}\,\rangle\pm|\,\mathbf{p}_1+\mathbf{q},\mathbf{p}_2-\mathbf{q}\,\rangle\Big).
\end{aligned}
$$

Again, the plus (minus) sign corresponds to bosons (fermions).

Exercise 12.2 Transition Amplitudes

We start by writing the three particle to three particle transition amplitude in terms of the operators $\hat{b}$ and $\hat{b}^\dagger$. We shorten notation by introducing the abbreviations $\hat{b}_{\mathbf{p}} \equiv \hat{b}(\mathbf{p}, t)$.

$$\begin{aligned}
&\langle\, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \,|\, \frac{1}{2}\hat{\mathcal{V}}_{\mathcal{F}} \,|\, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \,\rangle \\
&= \int \frac{\mathrm{d}^3 l_1\, \mathrm{d}^3 l_2\, \mathrm{d}^3 q}{\left(\sqrt{2\pi}\right)^3}\, \tilde{V}(\mathbf{q}) \left(\langle\, 0 \,|\, \hat{b}_{\mathbf{k}_3}\, \hat{b}_{\mathbf{k}_2}\, \hat{b}_{\mathbf{k}_1} \right) \hat{b}^\dagger_{\mathbf{l}_1}\, \hat{b}^\dagger_{\mathbf{l}_2}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}_{\mathbf{l}_1+\mathbf{q}} \left(\hat{b}^\dagger_{\mathbf{p}_1}\, \hat{b}^\dagger_{\mathbf{p}_2}\, \hat{b}^\dagger_{\mathbf{p}_3} \,|\, 0 \,\rangle \right)
\end{aligned} \tag{5}$$

The strategy is to use the commutation relations of the operators $\hat{b}$ and $\hat{b}^\dagger$ in order to move the annihilation operators to the right and the creation operators to the left. By use of the relation

$$\hat{b}_{\mathbf{p}}\, \hat{b}^\dagger_{\mathbf{p}'} = \delta^{(3)}(\mathbf{p} - \mathbf{p}') \pm \hat{b}^\dagger_{\mathbf{p}'}\, \hat{b}_{\mathbf{p}} \;, \tag{6}$$

every change of the order of an annihilation operator with a creation operator thus gives rise (apart from a minus sign for fermions) to an additional term where the two operators have annihilated themselves leading to a δ-funtion. In the end we want to use this procedure to eliminate all the operators leaving only terms where an annihilation operator is acting on the vacuum state $|\,0\,\rangle$. We will demonstrate one such step on the vacuum expectiation value by changing the order of the two operators $\hat{b}_{\mathbf{l}_1+\mathbf{q}}$ and $\hat{b}^\dagger_{\mathbf{p}_1}$ in Eq. (5)

$$\begin{aligned}
& \langle\, 0 \,|\, \hat{b}_{\mathbf{k}_3}\, \hat{b}_{\mathbf{k}_2}\, \hat{b}_{\mathbf{k}_1}\, \hat{b}^\dagger_{\mathbf{l}_1}\, \hat{b}^\dagger_{\mathbf{l}_2}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}_{\mathbf{l}_1+\mathbf{q}}\, \hat{b}^\dagger_{\mathbf{p}_1}\, \hat{b}^\dagger_{\mathbf{p}_2}\, \hat{b}^\dagger_{\mathbf{p}_3} \,|\, 0 \,\rangle \\
= \;& \langle\, 0 \,|\, \hat{b}_{\mathbf{k}_3}\, \hat{b}_{\mathbf{k}_2}\, \hat{b}_{\mathbf{k}_1}\, \hat{b}^\dagger_{\mathbf{l}_1}\, \hat{b}^\dagger_{\mathbf{l}_2}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}^\dagger_{\mathbf{p}_2}\, \hat{b}^\dagger_{\mathbf{p}_3} \,|\, 0 \,\rangle\, \delta^{(3)}(\mathbf{l}_1 + \mathbf{q} - \mathbf{p}_1) \\
& \pm \langle\, 0 \,|\, \hat{b}_{\mathbf{k}_3}\, \hat{b}_{\mathbf{k}_2}\, \hat{b}_{\mathbf{k}_1}\, \hat{b}^\dagger_{\mathbf{l}_1}\, \hat{b}^\dagger_{\mathbf{l}_2}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}^\dagger_{\mathbf{p}_1}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}^\dagger_{\mathbf{p}_2}\, \hat{b}^\dagger_{\mathbf{p}_3} \,|\, 0 \,\rangle
\end{aligned} \tag{7}$$

Keeping only the δ-function, we say that we have "contracted" the operators $\hat{b}_{\mathbf{l}_1+\mathbf{q}}$ and $\hat{b}^\dagger_{\mathbf{p}_1}$ and denote this by a bracket connecting the two operators,

$$\langle\, 0 \,|\, \hat{b}_{\mathbf{k}_3}\, \hat{b}_{\mathbf{k}_2}\, \hat{b}_{\mathbf{k}_1}\, \hat{b}^\dagger_{\mathbf{l}_1}\, \hat{b}^\dagger_{\mathbf{l}_2}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}_{\mathbf{l}_1+\mathbf{q}}\, \hat{b}^\dagger_{\mathbf{p}_1}\, \hat{b}^\dagger_{\mathbf{p}_2}\, \hat{b}^\dagger_{\mathbf{p}_3} \,|\, 0 \,\rangle = \langle\, 0 \,|\, \hat{b}_{\mathbf{k}_3}\, \hat{b}_{\mathbf{k}_2}\, \hat{b}_{\mathbf{k}_1}\, \hat{b}^\dagger_{\mathbf{l}_1}\, \hat{b}^\dagger_{\mathbf{l}_2}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}^\dagger_{\mathbf{p}_2}\, \hat{b}^\dagger_{\mathbf{p}_3} \,|\, 0 \,\rangle\, \delta^{(3)}(\mathbf{l}_1 + \mathbf{q} - \mathbf{p}_1). \tag{8}$$

Since commutating a destruction operator to the right such that it destroys the vacuum state leads to 0, the total vacuum expectation value is given by all possible contractions of destruction operators with a creation operator to their right, e.g. terms of the form,

$$\begin{aligned}
& \langle\, 0 \,|\, \hat{b}_{\mathbf{k}_3}\, \hat{b}_{\mathbf{k}_2}\, \hat{b}_{\mathbf{k}_1}\, \hat{b}^\dagger_{\mathbf{l}_1}\, \hat{b}^\dagger_{\mathbf{l}_2}\, \hat{b}_{\mathbf{l}_2-\mathbf{q}}\, \hat{b}_{\mathbf{l}_1+\mathbf{q}}\, \hat{b}^\dagger_{\mathbf{p}_1}\, \hat{b}^\dagger_{\mathbf{p}_2}\, \hat{b}^\dagger_{\mathbf{p}_3} \,|\, 0 \,\rangle \\
= \;& \delta(\mathbf{l}_1 + \mathbf{q} - \mathbf{p}_1)\, \delta(\mathbf{l}_2 - \mathbf{q} - \mathbf{p}_2)\, \delta(\mathbf{l}_1 - \mathbf{k}_1)\, \delta(\mathbf{l}_2 - \mathbf{k}_2)\, \delta(\mathbf{k}_3 - \mathbf{p}_3).
\end{aligned} \tag{9}$$

The expectation value in (7) is then simply a sum over terms of this form that can now be obtained by simply permuting the three creation operators belonging to the ket-state and the three destruction operators of the bra-state independently, giving rise to $36 = 3! \times 3!$ terms in the sum. Since fermions pick up a minus sign for every commutation performed, we additionally find a factor $(-1)^{p_1}(-1)^{p_2}$ for every term with $p_1, p_2 \in S_3$, permutations of three elements.

After performing the integrations, the final result can thus be written in a compact form as

$$\sum_{p_1, p_2 \in S_3} (\mp 1)^{p_1} (\mp 1)^{p_2}\, \tilde{V}(\mathbf{k}_{p_2(2)} - \mathbf{p}_{p_1(2)})\, \delta(\mathbf{k}_{p_2(1)} + \mathbf{k}_{p_2(2)} - \mathbf{p}_{p_1(2)} - \mathbf{p}_{p_1(1)})\, \delta(\mathbf{k}_{p_2(3)} - \mathbf{p}_{p_1(3)}) \tag{10}$$

where the minus (plus) sign corresponds to fermions (bosons).
Obviously, this procedure is very cumbersome, especially if we now wanted to go on to four-particle states. It is thus not surprising that better techniques to calculate these kinds of expectation values were invented, cf. Feynman diagrams.