Quantum Mechanics II
Solution Sheet 7

FS 09
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Exercise 7.1 Scattering cross section in the first Born approximation

a) As discussed in the lecture, the differential scattering cross section is directly related to the scattering amplitude via

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\mathbf{k},\mathbf{k}')|^2, \tag{1}$$

where $\mathbf{k}$ points in the direction of the incident beam and $\mathbf{k}'$ towards the observation point $\mathbf{r}$ (the solid angle Ω). For elastic scattering, we further know that $k = |\mathbf{k}| = |\mathbf{k}'|$.
In the first Born approximation, the scattering amplitude is (up to some numerical factor) given by the Fourier transform of the scattering potential with respect to $\mathbf{k}' - \mathbf{k}$,

$$f^{(1)}(\mathbf{k},\mathbf{k}') = -\frac{m}{2\pi\hbar^2}\widehat{V}(\mathbf{k}'-\mathbf{k}) = -\frac{m}{2\pi\hbar^2}\int d^3r'\, V(\mathbf{r}')e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}. \tag{2}$$

For a spherically symmetric potential, $f^{(1)}(\mathbf{k},\mathbf{k}')$ only depends on the absolute value $q = |\mathbf{k}-\mathbf{k}'| = 2k\sin\theta'/2$, where $\theta' = \sphericalangle(\mathbf{k},\mathbf{k}')$. The angular integration can be performed immediately,

$$f^{(1)}(\mathbf{k},\mathbf{k}') = f^{(1)}(k,\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty dr'\, r'V(r')\sin qr'. \tag{3}$$

For the spherical-box potential the integration is cut off at $r = r_0$ and one finds,

$$\begin{aligned} f^{(1)}(k,\theta) &= \frac{2mV_0}{\hbar^2 q}\int_0^{r_0} dr'\, r'\sin qr' \\ &= 2r_0\frac{mV_0r_0^2}{\hbar^2}\frac{\sin qr_0 - qr_0\cos qr_0}{(qr_0)^3}, \end{aligned} \tag{4}$$

and the differential scattering cross section reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f^{(1)}(\mathbf{k},\mathbf{k}')|^2 = 4r_0^2\Big(\frac{mV_0r_0^2}{\hbar^2}\Big)^2\frac{(\sin qr_0 - qr_0\cos qr_0)^2}{(qr_0)^6}. \tag{5}$$

b) The limit $kr_0 \to 0$ implies $qr_0 \to 0$. Expanding Eq. (5) in small qr_0 leads to an expression which is independent of q,

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{kr_0\to 0} &= 4r_0^2\Big(\frac{mV_0r_0^2}{\hbar^2}\Big)^2\frac{[qr_0 - (qr_0)^3/3! - qr_0 + (qr_0)^3/2! + \mathcal{O}((qr_0)^5)]^2}{(qr_0)^6} \\ &\approx \frac{4r_0^2}{9}\Big(\frac{mV_0r_0^2}{\hbar^2}\Big)^2. \end{aligned} \tag{6}$$

Since this expression is isotropic the total differential cross section at low energy is easily obtained,

$$\sigma = \int d\Omega\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{16\pi r_0^2}{9}\Big(\frac{mV_0r_0^2}{\hbar^2}\Big)^2. \tag{7}$$

c) For the Born approximation to be applicable the actual wave function $\Psi_k(\mathbf{r})$ should not be too different from the plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ inside the range of the scattering potential. In other words, the first Born approximation is reasonable if the following condition is fulfilled at the scattering center $\mathbf{r} = 0$ (where we assume the influence of the potential to be strongest),

$$\Big|\int d^3r'\, G(\mathbf{r}=0,\mathbf{r}')V(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'}\Big| = \Big|\frac{m}{2\pi\hbar^2}\int d^3r'\,\frac{e^{iqr'}}{r'}V(r')\Big| \ll 1, \tag{8}$$

where in the first equality we have used that $V(\mathbf{r}') = V(r')$. In the limit $kr_0 \to 0$ $(qr_0 \to 0)$ we can set $e^{iqr} \approx 1$ and one finds that the condition (8) holds if $V_0 m r_0^2/\hbar^2 \ll 1$.

Exercise 7.2 Low energy resonances and the Breit-Wigner formula

a) The continuity of the wave function and its first derivative implies

$$\partial_r \log R_l^{>}\big|_{r=R} = \partial_r \log R_l^{<}\big|_{r=R} = \alpha_l, \tag{9}$$

where the last equation is the definition of α_l. With the ansatz for the wave function outside the range R of the potential,

$$R_l^{>}(k,r) = \frac{1}{2}\left(h_l^*(kr) + e^{2i\delta_l(k)}h_l(kr)\right), \tag{10}$$

it follows that

$$\begin{aligned}\alpha_l &= \frac{k[\partial_x h_l^*(x) + e^{2i\delta_l(k)}\partial_x h_l(x))}{h_l^*(x) + e^{2i\delta_l(k)}h_l(x)}\Big|_{x=kR}\\ &= \frac{k[j_l'(x)(1+e^{2i\delta_l(k)}) + in_l'(x)(-1+e^{2i\delta_l(k)})]}{j_l(x)(1+e^{2i\delta_l(k)}) + in_l(x)(-1+e^{2i\delta_l(k)})}\Big|_{x=kR}\\ &= \frac{k[j_l'(x)\cos\delta_l(k) - n_l'(x)\sin\delta_l(k)]}{j_l(x)\cos\delta_l(k) - n_l(x)\sin\delta_l(k)}\Big|_{x=kR}\\ &= \frac{k[j_l'(x)\cot\delta_l(k) - n_l'(x)]}{j_l(x)\cot\delta_l(k) - n_l(x)}\Big|_{x=kR},\end{aligned} \tag{11}$$

where the prime stands for the derivative with respect to $x = kr$. Solving this for $\cot\delta_l(k)$ yields the equation

$$\cot\delta_l = \frac{k\partial_x n_l(x) - \alpha_l n_l(x)}{k\partial_x j_l(x) - \alpha_l j_l(x)}\Big|_{x=kR}. \tag{12}$$

Thus the phase shift depends on the potential only through the logarithmic derivative α_l.

b) We observe that

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \tag{13}$$

$$= \frac{4\pi}{k^2}(2l+1)\frac{\sin^2\delta_l}{\sin^2\delta_l + \cos^2\delta_l} \tag{14}$$

$$= \frac{4\pi}{k^2}(2l+1)\frac{1}{\cot^2\delta_l + 1}. \tag{15}$$

In order to find the maximum of $\sigma_l(k)$ in the limit $kR \ll 1$ we need to find the expression of Eq. (12) at low energies. With the asymptotic identities for the Bessel and Neumann functions,

$$j_l(x) \approx \frac{x^l}{(2l+1)!!}, \; n_l(x) \approx -\frac{(2l-1)!!}{x^{l+1}}, \tag{16}$$

as $x \longrightarrow 0$, one arrives at

$$\begin{aligned} \cot \delta_l &= \left. \frac{k\partial_x n_l(x) - \alpha_l n_l(x)}{k\partial_x j_l(x) - \alpha_l j_l(x)} \right|_{x=kR} \\ &\approx (2l-1)!!(2l+1)!! \, (kR)^{-(2l+1)} \, \frac{l+1+\alpha_l R}{l-\alpha_l R}. \end{aligned} \tag{17}$$

Inserting this into Eq. (15) leads to the expression

$$\sigma_l \approx \frac{4\pi}{k^2}(2l+1)\frac{1}{\left[(2l-1)!!(2l+1)!! \, (kR)^{-(2l+1)} \, (l+1+\alpha_l R)/(l-\alpha_l R)\right]^2 + 1} \tag{18}$$

$$\approx 4\pi R^2(2l+1)\frac{(kR)^{4l}}{\left[(2l-1)!!(2l+1)!! \, (l+1+\alpha_l R)/(l-\alpha_l R)\right]^2}, \tag{19}$$

which is proportional to $(kR)^{4l}$. Thus, at low energies the partial scattering cross section is a monotonically decreasing function of the angular quantum number l. In particular, its maximum is assumed for $l = 0$, and to a good approximation one can write

$$\sigma = \sum_{l \geq 0} \sigma_l \approx \sigma_{l=0} = 4\pi R^2 \Big(\frac{\alpha_0 R}{1+\alpha_0 R}\Big)^2. \tag{20}$$

Note, however, that there is an exception for which the step from (18) to (19) is not true; namely, if the particle energy $E_r = \hbar^2 k_r^2/2m$ is such that the following condition is fulfilled,

$$l + 1 + \alpha_l(E_r)R = 0. \tag{21}$$

In this case $\sigma_l \propto k^{-2}$ for *all* $l = 0, 1, 2, ...$ (recall that $kR \ll 1$). These *resonance energies* E_r maximize the partial scattering cross section σ_l for any channel l of the angular momentum.

c) Close to a resonance energy E_r one finds to leading order

$$\alpha_l(E)R \approx -(l+1) + \left. \frac{\mathrm{d}\alpha_l(E)R}{\mathrm{d}E} \right|_{E=E_r} (E - E_r). \tag{22}$$

We insert this into (17) to find

$$\begin{aligned} \cot \delta_l(E) &\approx (2l-1)!!(2l+1)!! \, (kR)^{-(2l+1)} \, \frac{\left.\frac{\mathrm{d}\alpha_l(E)R}{\mathrm{d}E}\right|_{E=E_r}(E-E_r)}{2l+1} \\ &= -\frac{2(E-E_r)}{\Gamma}, \end{aligned} \tag{23}$$

where Γ is defined as

$$\Gamma := -\frac{2k_r^{2l+1}R^{2l}}{[(2l-1)!!]^2 \left.\frac{\mathrm{d}\alpha_l(E)}{\mathrm{d}E}\right|_{E=E_r}}. \tag{24}$$

Thus, for scattering energies $E \approx E_r$ the cross section of the l-th channel is well-described by a Lorentzian,

$$
\begin{aligned}
\sigma_l &= \frac{4\pi}{k^2}(2l+1)\frac{1}{\cot^2\delta_l + 1} \\
&\approx \frac{4\pi}{k^2}(2l+1)\frac{1}{\frac{4(E-E_r)^2}{\Gamma^2} + 1} \\
&= \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma/2)^2}{(E-E_r)^2 + (\Gamma/2)^2},
\end{aligned} \tag{25}
$$

where the parameter Γ represents the width of the resonance. This result is called the Breit-Wigner formula.