Quantum Mechanics II
Solution Sheet 6

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Exercise 6.1 The one-dimensional Lippmann-Schwinger equation

a) The time-independent Schrödinger equation of the one-dimensional scattering problem reads

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\psi_k(x)=E_k\psi_k(x). \tag{1}$$

It is thus equivalent to

$$\left(\frac{d^2}{dx^2}+k^2\right)\psi_k(x)=\frac{2m}{\hbar^2}V(x)\psi_k(x) \tag{2}$$

if we define $k:=\sqrt{2mE}/\hbar$. Equation (2) is equivalent to the integral equation

$$\psi_k(x)=\psi_k^{(0)}(x)+\int_{\mathbb{R}}dy\,G_k(x-y)\frac{2m}{\hbar^2}V(y)\psi_k(y) \tag{3}$$

if one choses an appropriate Green's function $G_k(x)$. Recall that there is no unique Green's function associated to the differential operator on the LHS of equation (2). Consequently, once we have found a Green's function (i.e., a "function" G(x) which satisfies

$$\left(\frac{d^2}{dx^2}+k^2\right)G(x)=\delta(x) \tag{4}$$

in the sense of distributions) we have to check that it satisfies our demand

$$\psi_k(x)\sim e^{ikx}+f(k,k')\frac{e^{ik|x|}}{|x|} \tag{5}$$

($k':=k\,\mathrm{sgn}(x)$) as $x\to\pm\infty$. The wave function $\psi_k^{(0)}(x)$ is a solution of the homogenous version of equation (2) (i.e., RHS = 0). We chose $\psi_k^{(0)}(x)=e^{ikx}$ as in the lecture. For the purpose finding of the right Green's function we express $G_k(x)$ by its Fourier transform,

$$G_k(x)=\frac{1}{2\pi}\int_{\mathbb{R}}dq\,\hat{G}_k(q)e^{iqx} \tag{6}$$

and apply the operator $(d/dx)^2+k^2$

$$\begin{aligned}\left(\frac{d^2}{dx^2}+k^2\right)G_k(x)&=\frac{1}{2\pi}\int_{\mathbb{R}}dq\,\left(-q^2+k^2\right)\hat{G}_k(q)e^{iqx}\\&=\delta(x)=\frac{1}{2\pi}\int_{\mathbb{R}}dq\,e^{iqx}\end{aligned} \tag{7}$$

(recall equation (4)[1]). We can satisfy equation (7) if we set

$$\hat{G}_k(q)=\frac{1}{-q^2+k^2}. \tag{8}$$

Consequently, we have to evaluate the inverse Fourier transformation of $\hat{G}_k(q)$ which leads to the integral

$$G_k(x)=\frac{1}{2\pi}\int_{\mathbb{R}}dq\,\frac{e^{iqx}}{(k-q)(k+q)} \tag{9}$$

[1]Warning: the following argumentation might be confusing from a mathematical point of view. Please don't be confused about it! It's how it's usually done in textbooks.

which is not well defined because of the poles on the real axis. Consequently, we need to specify the meaning of the integral in (9). This is done as follows: we complexify q ($q \in \mathbb{R} \mapsto z \in \mathbb{C}$) and relate the unspecified integral to the limit of a family of integrals along paths $\{\gamma_\varepsilon\}_\varepsilon$ in the complex plane. In the end we have to check that the resulting function $G_k(x)$ really is a Green's function that satisfies our demand (5). It turns out that the choice

$$\gamma_\varepsilon := \lim_{R\to+\infty} C_1 \cup C_{\varepsilon,1} \cup C_2 \cup C_{\varepsilon,2} \cup C_3 \tag{10}$$

with

$$\begin{aligned}
C_{\varepsilon,1} &:= \{z = -k + \varepsilon e^{-i\theta} \;:\; \theta \in [0,\pi)\} \\
C_{\varepsilon,2} &:= \{z = k + \varepsilon e^{i\theta} \;:\; \theta \in [\pi, 2\pi)\} \\
C_1 &:= [-R, -k-\varepsilon) \\
C_2 &:= [-k+\varepsilon, k-\varepsilon) \\
C_3 &:= [k+\varepsilon, R)
\end{aligned}$$

leads to physically meaningful results. We thus make the ansatz

$$G_k(x) = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\gamma_\varepsilon} dz\, \frac{e^{izx}}{(k-z)(k+z)}. \tag{11}$$

As usual we use the residue theorem to compute this integral. For that purpose we close the path $C_1 \cup C_{\varepsilon,1} \cup C_2 \cup C_{\varepsilon,2} \cup C_3$ with the upper half circle

$$C_R^+ := \{z = Re^{i\theta} \;:\; \theta \in [0,\pi)\}$$

if $x > 0$ and with the lower half circle

$$C_R^- := \{z = Re^{i\theta} \;:\; \theta \in [\pi, 2\pi)\}$$

if $x < 0$. We call the resulting closed contour $\gamma^+_{\varepsilon,R}$ and $\gamma^-_{\varepsilon,R}$, respetively. We conclude that

$$G_k(x) = \lim_{\varepsilon\to 0} \lim_{R\to\infty} \frac{1}{2\pi} \left[\oint_{\gamma^\sharp_{\varepsilon,R}} \dots - \int_{C^\sharp_R} \dots \right] \tag{12}$$

where $\sharp$ is "+" if $x > 0$ and "-" if $x < 0$. The application of the residue theorem yields

$$\oint_{\gamma^\sharp_{\varepsilon,R}} \dots = -2\pi i \,\mathrm{Res}_{z=k}(\dots) = -2\pi i\, \frac{e^{ik|x|}}{2k} \tag{13}$$

independent of $\sharp \in \{+,-\}$. The C_R-integral vanishes according to Jordan's lemma as $R \to \infty$. Therefore we get

$$G_k(x) = -i\frac{e^{ik|x|}}{2k}. \tag{14}$$

and the Lippmann-Schwinger equation becomes

$$\psi_k(x) = e^{ikx} - \frac{2im}{\hbar^2} \int dy \frac{e^{ik|x-y|}}{2k} V(y)\psi_k(y). \tag{15}$$

b) With the attractive delta-potential,

$$\begin{aligned}
\psi_k(x) &= e^{ikx} + \frac{2im}{\hbar^2} \frac{e^{ik|x|}}{2k} \frac{\gamma\hbar^2}{2m} \psi_k(0) && (16)\\
&= e^{ikx} + \frac{i\gamma}{2k} e^{ik|x|} \psi_k(0) && (17)
\end{aligned}$$

This last equation can be solved for $\psi_k(0)$ if we set $x = 0$. We immediately get

$$\psi_k(0) = \frac{2k}{2k - i\gamma}. \tag{18}$$

We can consider the asymptotic forms (large $|x|$) of the above equation to obtain the transmission and reflection amplitudes. First we look at the case where $x \to \infty$. This gives

$$\psi_k(x) = e^{ikx} + \frac{i\gamma}{2k} e^{ikx} \psi_k(0) = e^{ikx} \frac{2k}{2k - i\gamma} = T(k) e^{ikx}. \tag{19}$$

For $x \to -\infty$ we deduce that

$$\psi_k(x) = e^{ikx} + \frac{i\gamma}{2k} e^{-ikx} \psi_k(0) = e^{ikx} + \frac{i\gamma}{2k - i\gamma} e^{-ikx} = e^{ikx} + R(k) e^{-ikx} \tag{20}$$

It immediately follows that $|T(k)|^2 + |R(k)|^2 = 1$ for all $k \in \mathbb{R}$.