Quantum Mechanics II
Solution Sheet 5

FS 09
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Exercise 5.1 Slow Turn-On of Perturbation

a) Up to first order the probability for the transition $|i\rangle \to |f\rangle$ is given by

$$P_{i\to f} = \frac{1}{\hbar^2}\left|\int_0^t dt'\, e^{\frac{i}{\hbar}(E_f-E_i)t'}\langle\psi_f, V(t')\psi_i\rangle\right|^2. \tag{1}$$

Now we define

$$\omega_{fi} := \frac{E_f - E_i}{\hbar}$$

and

$$V_{fi} := \langle\psi_f, V\psi_i\rangle,\ V_{fi}^\dagger := \langle\psi_f, V^\dagger\psi_i\rangle$$

to compute the following:

$$\begin{aligned}
P_{i\to f}(t) &= \frac{1}{\hbar^2}\left|\int_0^t dt'\, e^{i(\omega_{fi}+\omega)t'}V_{fi} + e^{i(\omega_{fi}-\omega)t'}V_{fi}^\dagger\right|^2 \\
&= \frac{1}{\hbar^2}\left|V_{fi}\,\frac{e^{i(\omega_{fi}+\omega)t'}}{i(\omega_{fi}+\omega)}\bigg|_0^t + V_{fi}^\dagger\,\frac{e^{i(\omega_{fi}-\omega)t'}}{i(\omega_{fi}-\omega)}\bigg|_0^t\right|^2 \\
&= \frac{1}{\hbar^2}\left|\frac{-iV_{fi}}{\omega_{fi}+\omega}e^{i\frac{\omega_{fi}+\omega}{2}t}2i\sin\left(\frac{\omega_{fi}+\omega}{2}t\right) - \frac{iV_{fi}^\dagger}{\omega_{fi}-\omega}e^{i\frac{\omega_{fi}-\omega}{2}t}2i\sin\left(\frac{\omega_{fi}-\omega}{2}t\right)\right|^2 \\
&= \frac{1}{\hbar^2}\frac{4|V_{fi}|^2}{(\omega_{fi}+\omega)^2}\sin^2\left(\frac{\omega_{fi}+\omega}{2}t\right) + &(2)\\
&\quad + \frac{1}{\hbar^2}\frac{4V_{fi}\overline{V_{fi}^\dagger}}{\omega_{fi}^2-\omega^2}\sin\left(\frac{\omega_{fi}+\omega}{2}t\right)\sin\left(\frac{\omega_{fi}-\omega}{2}t\right)e^{i\omega t} &(3)\\
&\quad + \frac{1}{\hbar^2}\frac{4\overline{V_{fi}}V_{fi}^\dagger}{\omega_{fi}^2-\omega^2}\sin\left(\frac{\omega_{fi}+\omega}{2}t\right)\sin\left(\frac{\omega_{fi}-\omega}{2}t\right)e^{-i\omega t} &(4)\\
&\quad + \frac{1}{\hbar^2}\frac{4|V_{fi}^\dagger|^2}{(\omega_{fi}-\omega)^2}\sin^2\left(\frac{\omega_{fi}-\omega}{2}t\right) &(5)
\end{aligned}$$

The transition probability to a set $\{f\}$ of final states is

$$P_{i\to\{f\}}(t) = \sum_{f\in\{f\}} P_{i\to f}(t). \tag{6}$$

The continuum limit of this last equation reads (for transitions to anywhere in the continuous spectrum)

$$\begin{aligned}
P_{i\to\{f\}}(t) &= \sum_{E_f\in\{E_j : j\in\{f\}\}} P_{i\to f}\cdot \text{Degeneracy}(E_f) &(7)\\
&= \sum_{E_f\in\{E_j : j\in\{f\}\}} [P_{i\to f}\rho](E_f)\cdot(E_{f+1}-E_f) &(8)\\
&\to \int dE_f\, [P_{i\to f}\rho](E_f), &(9)
\end{aligned}$$

as $\sup_j\{E_{j+1} - E_j\} \to 0$. Here we have defined the spectral density $\rho(E)$ in the discrete case such that

$$\text{Degeneracy}(E_f) = \rho(E_f)(E_{f+1} - E_f).$$

We thus get

$$\Gamma_{i\to\{f\}} = \frac{dP_{i\to\{f\}}(t)}{dt} = \int dE_f\, \rho(E_f)\, \frac{dP_{i\to f}(t)}{dt} \tag{10}$$

for the transition rate to any "state" in the continuum. Next we use our result (2) - (5) for $P_{i\to f}(t)$ and assume that the contributions (3), (4) vanish after the evaluation of the integral in (10) for large t (fast oscillations which average to zero):

$$\begin{aligned}
\Gamma_{i\to\{f\}} &= \frac{4}{\hbar^2}\int_{\mathbb{R}} dE_f\, \rho(E_f)\Big[\frac{|V_{fi}|^2}{(\omega_{fi}+\omega)^2} 2\sin\left(\frac{\omega_{fi}+\omega}{2}t\right)\cos\left(\frac{\omega_{fi}+\omega}{2}t\right)\frac{\omega_{fi}+\omega}{2} \\
&\quad + \frac{|V_{fi}^\dagger|^2}{(\omega_{fi}-\omega)^2} 2\sin\left(\frac{\omega_{fi}-\omega}{2}t\right)\cos\left(\frac{\omega_{fi}-\omega}{2}t\right)\frac{\omega_{fi}-\omega}{2}\Big] &(11)\\
&= \frac{2}{\hbar^2}\int_{\mathbb{R}} dE_f\, \rho(E_f)\Big[|V_{fi}|^2\, \frac{\sin\left((\omega_{fi}+\omega)t\right)}{\omega_{fi}+\omega} + |V_{fi}^\dagger|^2\, \frac{\sin\left((\omega_{fi}-\omega)t\right)}{\omega_{fi}-\omega}\Big] &(12)\\
&= \frac{2}{\hbar^2}\int_{\mathbb{R}} dE_f\, \rho(E_f)\Big[|V_{fi}|^2\, \frac{\sin\left(\frac{E_f-E_i+\omega\hbar}{\hbar}t\right)}{\omega_{fi}+\omega} + |V_{fi}^\dagger|^2\, \frac{\sin\left(\frac{E_f-E_i-\omega\hbar}{\hbar}t\right)}{\omega_{fi}-\omega}\Big]. &(13)
\end{aligned}$$

Next we average the matrix elements over the continuous spectrum and put the resulting scalar (notation: $\langle|V_{fi}|^2\rangle$, $\langle|V_{fi}^\dagger|^2\rangle$) in front of the integral. This leads to an approximation of the exact result. The substitutions

$$E_f \mapsto u := \frac{E_f - E_i + \omega\hbar}{\hbar},\ E_f \mapsto v := \frac{E_f - E_i - \omega\hbar}{\hbar}$$

in the first respectively second term and the application of the formula

$$\lim_{x\to\infty} \frac{\sin[x(\omega-\omega_0)]}{\omega-\omega_0} = \pi\delta(\omega-\omega_0) \tag{14}$$

from exercise 5.3 give the result:

$$\begin{aligned}
\lim_{t\to\infty}\Gamma_{i\to\{f\}} &= \frac{2\langle|V_{fi}|^2\rangle}{\hbar}\pi\delta_0\big(\rho(\hbar u + E_i - \omega\hbar)\big) + \frac{2\langle|V_{fi}^\dagger|^2\rangle}{\hbar}\pi\delta_0\big(\rho(\hbar u + E_i + \omega\hbar)\big) &(15)\\
&= \frac{2\pi}{\hbar}\left(\langle|V_{fi}|^2\rangle\rho(E_i-\omega\hbar) + \langle|V_{fi}^\dagger|^2\rangle\rho(E_i+\omega\hbar)\right), &(16)
\end{aligned}$$

or written in terms of this distribution this reads

$$\lim_{t\to\infty}\Gamma_{i\to\{f\}}(\rho) = \frac{2\pi}{\hbar}\left(\langle|V_{fi}|^2\rangle\, \delta_{E_i-\omega\hbar}(\rho) + \langle|V_{fi}^\dagger|^2\rangle\, \delta_{E_i+\omega\hbar}(\rho)\right) \tag{17}$$

for $\rho \in \mathcal{S}(\mathbb{R})$.

b) You start with the equality

$$P_{i\to f} = \lim_{t_0\to-\infty}\frac{1}{\hbar^2}\left|\int_{t_0}^{t} dt'\, e^{\frac{i}{\hbar}(E_f-E_i)(t'-t_0)}\langle\psi_f, V(t')\psi_i\rangle\right|^2. \tag{18}$$

Afterwards the calculation is absolutely similar to the calculation in 5.3.a). In the end you get

$$\Gamma_{i\to\{f\}} = \frac{2\pi}{\hbar}\Big(\langle|V_{fi}|^2\rangle\rho(E_i - \omega\hbar - \hbar\delta) + \langle|V_{fi}^\dagger|^2\rangle\rho(E_i + \omega\hbar - \hbar\delta)\Big) \tag{19}$$

or written in terms of this distribution this reads

$$\lim_{t\to\infty}\Gamma_{i\to\{f\}}(\rho) = \frac{2\pi}{\hbar}\Big(\langle|V_{fi}|^2\rangle\,\delta_{E_i-\omega\hbar-\hbar\delta}(\rho) + \langle|V_{fi}^\dagger|^2\rangle\,\delta_{E_i+\omega\hbar-\hbar\delta}(\rho)\Big)\,. \tag{20}$$

Note that the slow turn on of the potential has led to a shift $-\hbar\delta$.

Exercise 5.2 Sudden Constant Perturbation

a) We want to prove that the density of states $\rho(E_f)$ is normalized, i.e.

$$\int_{-\infty}^{\infty}\rho(E_f)\,\mathrm{d}E_f = 1\,, \tag{21}$$

$$\rho(E_f) = \frac{1}{\pi}\frac{1}{1+E_f^2}\,. \tag{22}$$

The integrand corresponds (up to a factor of π^{-1}) to the derivative of the arcus tangent, and thus we can write

$$\int_{-\infty}^{\infty}\rho(E_f)\,\mathrm{d}E_f = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}E_f}\Big(\arctan(E_f)\Big)\,\mathrm{d}E_f \tag{23}$$

$$= \frac{1}{\pi}\lim_{x\to\infty}\Big[\arctan x - \arctan -x\Big] \tag{24}$$

$$= \frac{1}{\pi}\Big[\frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big)\Big] = 1 \tag{25}$$

b) Here we are interested in the transition of a system set up in a certain *normalized* state $|i\rangle$ into a continuum of states $|f\rangle$. Note that in the case of a continuum, a single state $|f\rangle$ is not a physical state due to its non-normalizability. If we consider a physical situation, we must always integrate over a set of states weighted by a probability distribution. This is well known for the case of *physical* wave-packets constructed from *unphysical* plane waves.

In this exercise, we consider the transition into the entire continuum of state. In order to get the transition probability (or transition rate) we have to "sum" the probability (or rate) of transition into the individual states of the continuum.

According to the formalism of time-dependent perturbation theory, we know that the (differential) probability $\delta\mathcal{P}$ for the transition from the state $|i\rangle$ into the state $|f\rangle$ is given by

$$\delta\mathcal{P}_{i\to f} = \frac{1}{\hbar^2}\,|V_{fi}|^2\,\frac{\sin^2(\omega_{fi}t/2)}{(\omega_{fi}/2)^2}\,. \tag{26}$$

Here we have already used the information that we consider a potential of the form $V(t) = V\Theta(t)$.

To get a physical probability we must integrate the differential probability over the set of final states $\{f\}$ we are interested in. Usually it is convenient to label the states by their energy and weight the integrand with the "states per energy", the so-called density of states $\rho(E_f)$. The integration is then taken over the energy interval $\mathcal{D}$ where the states of interest are located,

$$\mathcal{P}_{i\to\{f\}} = \int_{\mathcal{D}} \mathrm{d}E_f\, \rho(E_f)\, \delta\mathcal{P}_{i\to f}\ . \tag{27}$$

Here we take $\{f\}$ to be the entire continuum and find

$$\mathcal{P}_{i\to\{f\}} = \frac{|\lambda|^2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}E_f\, \frac{1}{1+E_f^2}\, \frac{\sin^2\big((E_f-1)\,t\,/\,2\hbar\big)}{(E_f-1)^2/4}\ , \tag{28}$$

where we have used the informaltion that $|V_{fi}|^2$ is given by $|\lambda|^2$.

We could now perform the (complicated) integration directly, but it turns out that it is convenient to first perform a time derivative to obtain the transition rate $\Gamma_{i\to\{f\}}$, defined by

$$\Gamma_{i\to\{f\}} = \frac{\mathrm{d}\,\mathcal{P}_{i\to\{f\}}}{\mathrm{d}t}\ . \tag{29}$$

After calculating the rate, we can get back to the probability by simple time integration. We find that the transition rate is defined by the following expression,

$$\begin{aligned}\Gamma_{i\to\{f\}} &= \frac{|\lambda|^2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}E_f\, \frac{1}{1+E_f^2}\, \frac{\mathrm{d}}{\mathrm{d}t}\, \frac{\sin^2\big((E_f-1)\,t\,/\,2\hbar\big)}{(E_f-1)^2/4} &(30)\\ &= \frac{2\,|\lambda|^2}{\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}E_f\, \frac{1}{1+E_f^2}\, \frac{\sin\big((E_f-1)\,t\,/\,\hbar\big)}{E_f-1} &(31)\end{aligned}$$

Now the integral that has to be solved is slightly less complicated but more important, it can be compared with the integral which has to be solved in Exercise 5.3. We will evaluate the integral with the residue theorem which is very useful when the integration contour can be closed by a contour where the integrand identically vanishes. For this we rewrite the sin as sum of exponential functions, which exponentially decay on a large half circle in either the upper or the lower complex half plane.

$$\Gamma_{i\to\{f\}} = \frac{|\lambda|^2}{i\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}E_f\, \frac{1}{1+E_f^2}\, \frac{e^{i(E_f-1)t/\hbar} - e^{-i(E_f-1)t/\hbar}}{E_f-1} \tag{32}$$

One cannot separate the integration into two different integrals without paying attention to the fact that the original sin removes the singular behavior of the $(E_f-1)^{-1}$ at $E_f \approx 1$. Because the integrand in (31) is completely analytic on the entire real axis we are free to deform the integration contour continuously. Hence, we choose to take a contour that runs on a small half circle in the upper complex plane around $E_f = 1$. Note that $E_f = 1$ the integrand (31) is regular and does not have a pole and thus we could have also chosen a half circle in the lower complex plane! Technically, this procedure can be conveniently performed by introducing an infinitesimal imaginary part in the denominator of $(E_f-1)^{-1}$,

$$\frac{1}{E_f-1} \to \frac{1}{E_f-1+i\epsilon}\ . \tag{33}$$

In summary, we can express the transition rate from state $|i\rangle$ into the whole continuum of states $|f\rangle$ as

$$\Gamma_{i\to\{f\}} = \lim_{\epsilon\to 0^+} \frac{|\lambda|^2}{i\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}E_f \, \frac{1}{1+E_f^2} \, \frac{e^{i(E_f-1)t/\hbar} - e^{-i(E_f-1)t/\hbar}}{E_f - 1 + i\epsilon} \,. \tag{34}$$

Now we can split up the integration into two parts that can be evaluated independently, because the integrations are independently well defined

$$\begin{aligned}\Gamma_{i\to\{f\}} = \lim_{\epsilon\to 0^+} \frac{|\lambda|^2}{i\pi\hbar} \Bigg\{ & \int_{-\infty}^{\infty} \mathrm{d}E_f \, \frac{1}{1+E_f^2} \, \frac{e^{i(E_f-1)t/\hbar}}{E_f - 1 + i\epsilon} \\ & - \int_{-\infty}^{\infty} \mathrm{d}E_f \, \frac{1}{1+E_f^2} \, \frac{e^{-i(E_f-1)t/\hbar}}{E_f - 1 + i\epsilon} \Bigg\} \,. \end{aligned} \tag{35}$$

Note that the limit cannot be performed independently for the two terms.

Now the independent integrals can be computed by the residue theorem. For the first (second) integral we close the integration contour in the upper (lower) complex plane on a large circle of radius that tends to infinity and thus gives no contribution due to exponential suppression. The integrals can then be identified with the residue of the poles inside the integration contour. Hence, we find

$$\begin{aligned}\Gamma_{i\to\{f\}} = \lim_{\epsilon\to 0^+} \frac{2\,|\lambda|^2}{\hbar} \Bigg\{ & \mathrm{Res}_{E_f=i} \left(\frac{1}{1+E_f^2} \, \frac{e^{i(E_f-1)t/\hbar}}{E_f - 1 + i\epsilon} \right) \\ & + \mathrm{Res}_{E_f=-i} \left(\frac{1}{1+E_f^2} \, \frac{e^{-i(E_f-1)t/\hbar}}{E_f - 1 + i\epsilon} \right) \\ & + \mathrm{Res}_{E_f=1-i\epsilon} \left(\frac{1}{1+E_f^2} \, \frac{e^{-i(E_f-1)t/\hbar}}{E_f - 1 + i\epsilon} \right) \Bigg\} \,. \end{aligned} \tag{36}$$

The first residue comes from the pole at $E_f = i$ of the density of states enclosed by the first integration contour. The second residue comes from the opposite pole of $\rho(E_f)$ at $E_f = -i$ in the lower complex plane while the last residue has its origin in the pole that we uncover by splitting the original integral into two parts and shifting this pole infinitesimally away from the real axis into the lower complex plane. Note that the two residue from the integration contour in the lower complex plane acquire a sign due to the clockwise direction of the integration contour. After computing the residue we find

$$\Gamma_{i\to\{f\}} = \lim_{\epsilon\to 0^+} \frac{2\,|\lambda|^2}{\hbar} \left\{ \left(\frac{e^{-\frac{t}{\hbar} - \frac{it}{\hbar}}}{2(-1-i-\epsilon)} \right) + \left(\frac{e^{-\frac{t}{\hbar} + \frac{it}{\hbar}}}{2(-1+i+\epsilon)} \right) + \left(\frac{e^{-\frac{\epsilon t}{\hbar}}}{2 - 2i\epsilon - \epsilon^2} \right) \right\} . \tag{37}$$

From this expression one can confirm that the result is as it should be real. The first two terms are complex conjugates and the imaginary parts of the last term vanish after taking the limit of $\epsilon \to 0$, which gives

$$\begin{aligned}\Gamma_{i\to\{f\}} &= \frac{|\lambda|^2}{\hbar} \left(\frac{1}{2} + \frac{i}{2} \right) \left(i e^{-\frac{(1+i)t}{\hbar}} - e^{-\frac{(1-i)t}{\hbar}} + (1-i) \right) \\ &= \frac{|\lambda|^2}{\hbar} \left(1 - e^{-t/\hbar} \left(\cos(t/\hbar) - \sin(t/\hbar) \right) \right) \end{aligned} \tag{38}$$

Interestingly, we find a transition rate that is time-dependent, but the time dependence also decays exponentially in time. Neglecting this exponentially suppressed term, the probability $\mathcal{P}_{i-\{f\}}$ is then found by (then trivial) time integration, which corresponds to a multiplication with the time t.

c) In the situation we consider in this exercise Fermi's Golden Rule (FGR) can be applied for the limit of large times[1]. Note that in order to deal with "large times" we technically perform the limit $t \to \infty$. Nevertheless, it should be kept in mind that perturbation theory, and in deed Fermi's Golden Rule conceptually corresponds to first order perturbation theory, breaks down for very large times. Here we do not want to give quantitate boundaries t for which FGR holds.

According to FGR, we get the rate for a transition from a state $|i\rangle$ into the entire continuum of states $|f\rangle$ as

$$\Gamma^{\text{FGR}}_{i\to\{f\}} = \frac{2\pi}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}E_f \rho(E_f)|\lambda|^2\delta(E_f - 1) = \frac{2\pi}{\hbar}|\lambda|^2\rho(1) = \frac{|\lambda|^2}{\hbar}\,. \tag{39}$$

In order to compare this result to our calculation of part (b), we take the limit $t \to \infty$ of Eq. (38), we arrive at the asymptotic expression for the transition rate,

$$\lim_{t\to\infty} \Gamma_{i\to\{f\}} = \frac{|\lambda|^2}{\hbar}\,, \tag{40}$$

which can be identified with the result of Fermi's Golden Rule, Eq. (39).

As mentioned already in the end of part (b) of this exercise, the time dependence is only given by an exponentially suppressed (in time) term. In the limit of large times, this terms drops out very quickly and the results $\Gamma^{\text{FGR}}_{i\to\{f\}}$ and (38) coincide. This result is also consistent with the fact that we do not really want to consider the limiting case of $t = \infty$ but rather "intermediate large" times.

Exercise 5.3 The Delta Function

We want to show that in the sense of distributions the following equality holds

$$\lim_{x\to\infty} \frac{\sin[(\omega-\omega_0)x]}{\omega-\omega_0} = \pi\delta(\omega-\omega_0). \tag{41}$$

In other words, we need to make sure that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} d\omega\, \rho(\omega)\frac{\sin[(\omega-\omega_0)x]}{\omega-\omega_0} = \pi\rho(\omega_0), \tag{42}$$

for any test-function $\rho \in \mathcal{S}(\mathbb{R})$, the Schwartz space over $\mathbb{R}$. The elements of $\mathcal{S}(\mathbb{R})$ are bounded on $\mathbb{R}$ and in particular, they don't have a pole. Hence, the integrand on the left hand side of Eq. (42) is analytic on the real axis. Note that $\sin x/x \to 1$ for $x \to 0$ (i.e.,

[1] It is a topic of recent research under exactly which conditions FGR can be actually applied. The problem is that in the mathematical derivation these conditions are much more restrictive than in real life. Fermi's Golden Rule is called "Golden Rule" because it is one of the most important relations in an extremely wide range of experimental physics.

$x = 0$ is a removable singularity).
We may thus write

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega\, \rho(\omega) \frac{\sin[(\omega-\omega_0)x]}{\omega-\omega_0} &= \int_{-\infty}^{\infty} d\omega\, \rho(\omega) \lim_{\epsilon\to 0^+} \frac{\sin[(\omega-\omega_0)x]}{\omega-\omega_0+i\epsilon} \\ &= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} d\omega\, \rho(\omega) \frac{\sin[(\omega-\omega_0)x]}{\omega-\omega_0+i\epsilon} \\ &= \lim_{\epsilon\to 0^+} \left\{ \frac{1}{2i} \int_{-\infty}^{\infty} d\omega\, \rho(\omega) \frac{e^{i(\omega-\omega_0)x}}{\omega-\omega_0+i\epsilon} \right. \\ &\qquad \left. - \frac{1}{2i} \int_{-\infty}^{\infty} d\omega\, \rho(\omega) \frac{e^{-i(\omega-\omega_0)x}}{\omega-\omega_0+i\epsilon} \right\}. \end{aligned} \tag{43}$$

For the evaluation of these two integrals we make use of the residue theorem (and implicitly assume that the test-function ρ possesses an analytic continuation on the whole complex plane). With $x > 0$ (which surely holds in the limit $x \to \infty$), we close the contour with a semicircle of radius R in the positive (negative) complex half-plane for the first (second) integral and let R tend to ∞. We find

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega\, \rho(\omega) \frac{\sin[(\omega-\omega_0)x]}{\omega-\omega_0} =& \pi \lim_{\epsilon\to 0^+} \left\{ \sum_k \mathrm{Res}_{a_k} \left[\rho(\omega) \frac{e^{i(\omega-\omega_0)x}}{\omega-\omega_0+i\epsilon} \right] \right. \\ & \left. + \sum_k \mathrm{Res}_{b_k} \left[\rho(\omega) \frac{e^{-i(\omega-\omega_0)x}}{\omega-\omega_0+i\epsilon} \right] + \mathrm{Res}_{\omega_0 - i\epsilon} \left[\rho(\omega) \frac{e^{-i(\omega-\omega_0)x}}{\omega-\omega_0+i\epsilon} \right] \right\} \\ =& \pi \left\{ \sum_k \mathrm{Res}_{a_k} \left[\rho(\omega) \frac{e^{i(\omega-\omega_0)x}}{\omega-\omega_0} \right] \right. \\ & \left. + \sum_k \mathrm{Res}_{b_k} \left[\rho(\omega) \frac{e^{-i(\omega-\omega_0)x}}{\omega-\omega_0} \right] + \mathrm{Res}_{\omega_0} \left[\rho(\omega) \frac{e^{-i(\omega-\omega_0)x}}{\omega-\omega_0} \right] \right\}, \end{aligned} \tag{44}$$

where the sets $\{a_k\}$ and $\{b_k\}$ denote the poles of the function ρ in the upper and lower half-plane. In the limit $x \to \infty$, the residues of these poles vanish due to the exponential suppression $e^{i(a_k-\omega_0)x} \propto e^{-\mathrm{Im} a_k x}$ and $e^{-i(b_k-\omega_0)x} \propto e^{\mathrm{Im} b_k x}$, respectively. Note that $\mathrm{Im} a_k > 0$ and $\mathrm{Im} b_k < 0$ for all k. Hence, only the residue at ω_0 contributes. With

$$\mathrm{Res}_{\omega_0} \left[\rho(\omega) \frac{e^{-i(\omega-\omega_0)x}}{\omega-\omega_0} \right] = \rho(\omega_0), \tag{45}$$

we arrive at

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} d\omega\, \rho(\omega) \frac{\sin[(\omega-\omega_0)x]}{\omega-\omega_0} = \pi \rho(\omega_0). \tag{46}$$