Quantum Mechanics II
Exercise 3

FS 09
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Exercise 3.1 Degenerate Perturbation Theory III: The Stark Effect

We consider a hydrogen atom placed in an external homogeneous electric field $\mathbf{E}$. Thus, the Hamiltonian of the hydrogen atom is modified by the term

$$\delta H = e\,\mathbf{E}\cdot\mathbf{x}\ , \tag{1}$$

which describes the coupling of the external field to the electric dipole moment $e\mathbf{x}$ of the atom. In this exercise we can take the field to be directed along the z-axis.
For simplicity, we assume the field to be strong enough to neglect the fine and hyperfine splitting of the hydrogen levels, but sufficiently weak to treat δH perturbatively.

a) Try to estimate up to which order of magnitude in the electric field strength, a perturbative treatment of δH makes sense: Compare the external electric field with the field which is responsible for the energy levels of a free hydrogen atom. Is a perturbative treatment of δH reasonable for typical strong electric fields accessible in laboratories, e.g., the breakthrough voltage for a 1 cm thick piece of SiO_2, one of the best isolators(!), of magnitude $10^6\,\mathrm{V} - 10^8\,\mathrm{V}$?

b) Determine the change in the energy spectrum to first order perturbation theory in the perturbation δH for the hydrogen levels with $n = 2$.

   The wave functions for the $n = 2$ level of the hydrogen atom are

$$\Psi_{2,0,0} = \frac{1}{\sqrt{8\pi a_0^3}}\left(1 - \frac{r}{2a_0}\right)e^{-\frac{r}{2a_0}}\ , \tag{2}$$

$$\Psi_{2,1,m} = \frac{1}{\sqrt{8a_0^3}}\frac{r}{\sqrt{3}a_0}e^{-\frac{r}{2a_0}}Y_{1,m}(\theta,\phi)\ , \tag{3}$$

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta\ , \tag{4}$$

$$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta\ . \tag{5}$$

Exercise 3.2 Time-Dependent Perturbation of a Two-Level System

We consider a two-level system with energies $E_1 < E_2$,

$$H_0 = \sum_{i=1,2} E_i |i\rangle\langle i| \ . \tag{6}$$

The two levels are connected by a time-dependent potential $V(t)$ as follows:

$$V_{11} = V_{22} = 0 \ , \quad V_{12} = \gamma e^{i\omega t} \ , \quad V_{21} = \gamma e^{-i\omega t} \ , \tag{7}$$

where γ is a parameter expressed in units of energy.
We assume only the lower level to be populated at $t = 0$,

$$c_1(0) = 1 \ , \quad c_2(0) = 0 \ , \tag{8}$$

where $c_i(t)$ are the coefficients that determine the population of the state, i.e.

$$|\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle \ . \tag{9}$$

These coefficients satisfy the coupled differential equation

$$i\hbar \, \dot{c}_k(t) = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_{kn}t} c_n(t) \ , \quad (k = 1, 2) \ , \tag{10}$$

where

$$\omega_{kn} = \frac{(E_k - E_n)}{\hbar} = -\omega_{nk} \ . \tag{11}$$

a) Show that the relation

$$|c_1(t)|^2 + |c_2(t)|^2 = 1 \tag{12}$$

holds for all times.

b) Find the probabilities $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ by exactly solving the coupled differential equation (10).

c) Consider the same problem using time-dependent perturbation theory to first order.

d) Compare the two approaches for small values of γ, i.e. $E_2 - E_1 \gg \gamma$ (*why?*). Treat the following two cases separately:

(i) ω is very different from ω_{21}, i.e. $\gamma \ll \frac{|\omega - \omega_{21}|}{2}\hbar$

(ii) ω is very close to ω_{21}, i.e. $\gamma \gg \frac{|\omega - \omega_{21}|}{2}\hbar$