Quantum Information Theory
Serie 13

Exercise 13.1 Entanglement as a resource

Suppose Alice and Bob share a state ρ_{AB} on $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$. We wish to show that entanglement does not increase under local operations and classical communication (LOCC). We measure entanglement using

$$E(A:B) := \frac{1}{2} \min_{\rho_{\mathrm{ABR}}} I(A:B|R), \tag{1}$$

where the minimum is taken over all extensions ρ_{ABR} of ρ_{AB} such that $\mathrm{tr}_{\mathrm{R}}(\rho_{\mathrm{ABR}}) = \rho_{\mathrm{AB}}$.

a) Given a pure state ρ_{AB}, show that $\rho_{\mathrm{ABR}} = \rho_{\mathrm{AB}} \otimes \rho_{\mathrm{R}}$.

b) A maximally entangled state on $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$ is a pure state ρ_{AB} with $\mathrm{tr}_{\mathrm{A}}(\rho_{\mathrm{AB}}) = \mathbb{1}_{\mathrm{B}}/d_{\mathrm{B}}$ and $\mathrm{tr}_{\mathrm{B}}(\rho_{\mathrm{AB}}) = \mathbb{1}_{\mathrm{A}}/d_{\mathrm{A}}$, where $d_{\mathrm{A}} = \dim \mathcal{H}_{\mathrm{A}}$. Calculate $E(A:B)$ for the case when $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$ are two-dimensional.

We first show that local operations (LO) cannot increase entanglement. A local operation is a CPM and can be decomposed into three parts: (i) introduce of an extension $\rho_{\mathrm{ABR}} \mapsto \rho_{\mathrm{AA'BR}} = \rho_{\mathrm{ABR}} \otimes \sigma_{\mathrm{A'}}$, (ii) apply a unitary on $\mathcal{H}_{\mathrm{A}}$ and finally (iii) do a partial trace over some subsystem of $\mathcal{H}_{\mathrm{A}}$.

c) Show that extensions (i) and unitaries (ii) do not change mutual information $I(A:B|R)$.

d) Show that a partial trace (iii) can only reduce mutual information $I(A:B|R)$.

Note that we did not use the minimum in the definition of $E(A:B)$ so far.
Next we show that classical communication (CC) does not increase entanglement. Classical communication consists of a measurement on Alice's side followed by communication of the measurement result. Mathematically, Alice prepares a state that is classical in A' using local operations:

$$\rho_{\mathrm{AA'BR}} = \sum_i p_i \, |i\rangle\langle i|_{\mathrm{A'}} \otimes \rho^i_{\mathrm{ABR}}. \tag{2}$$

Classical communication maps this state to

$$\rho_{\mathrm{AA'BR}} \mapsto \rho_{\mathrm{AA'BB'R''}} = \sum_i p_i \, |i\rangle\langle i|_{\mathrm{A'}} \otimes |i\rangle\langle i|_{\mathrm{B'}} \otimes \rho^i_{\mathrm{ABR''}}, \tag{3}$$

where R'' is another extension of the system.

e) Show that $I(A:B)$ can increase under classical communication by giving an example. Hint: Start with an empty system and share a classical random variable.

f) Show that $I(A:B|R)$ does not increase in your example for an appropriate choice of extension R.

g) Show that for any R there exists an R' such that $I(A,A':B|R) \geq I(A,A':B,B'|R')$.