Quantum Information Theory
Serie 9

Exercise 9.1 Theory, Experiment and Paranoia

Let us consider the following thought experiment: Alice and Bob share a system that is completely described by a random variable Z (the hidden variable). They have a choice of two measurements $A \in \{A_1, A_2\}$ and $B \in \{B_1, B_2\}$ respectively that they can apply to their part of the system. The outputs will be $X \in \{-1, 1\}$ on Alice's side and $Y \in \{-1, 1\}$ on Bob's side. The measurement process can be modelled by a conditional probability distribution $P_{\mathrm{XY|ABZ}}$.
Let us now introduce two assumptions that could be made about this experiment: (i) the result is deterministic given the hidden variable: $P_{\mathrm{XY|ABZ}} \in \{0, 1\}$ and (ii) it is local: $P_{\mathrm{XY|ABZ}} = P_{\mathrm{X|AZ}} \cdot P_{\mathrm{Y|BZ}}$. The latter condition simply states that the measurement outcome on Alice's side does not depend on Bob's choice of measurement or on his measurement outcome.

a) Let us define $\mathcal{P}^{\neq}_{a,b}(z) := P_{\mathrm{XY|ABZ}}[X \neq Y | A = a, B = b, Z = z]$ and $\mathcal{P}^{=}_{a,b}(z) = 1 - \mathcal{P}^{\neq}_{a,b}(z)$ accordingly. Use locality (ii) and determinism (i) to derive the following bound:

$$\mathcal{P}^{\neq}_{\mathrm{A_1,B_1}}(z) + \mathcal{P}^{\neq}_{\mathrm{A_2,B_1}}(z) + \mathcal{P}^{\neq}_{\mathrm{A_2,B_2}}(z) + \mathcal{P}^{=}_{\mathrm{A_1,B_2}}(z) \geq 1. \tag{1}$$

Can it be done without assumption (i)?

b) Use (1) to derive Bell's inequality:

$$\langle XY \rangle_{\mathrm{A_1,B_1}} + \langle XY \rangle_{\mathrm{A_2,B_1}} + \langle XY \rangle_{\mathrm{A_2,B_2}} - \langle XY \rangle_{\mathrm{A_1,B_2}} \leq 2, \tag{2}$$

where $\langle XY \rangle_{ab} = \sum_z P_{\mathrm{Z}}(z) \, (\mathcal{P}^{=}_{a,b}(z) - \mathcal{P}^{\neq}_{a,b}(z))$ is the expectation value of the product $X \cdot Y$ over Z.

c) Show that quantum mechanics violates (1) by means of an example.

The paranoid experimentalist
Consider two identical imperfect detectors that correctly report $X = \pm 1$ with probability η and a "no click" event $X = \perp$ otherwise, in particular the detectors never give false positives. A Bell experiment is done with these detectors and you are given the task to evaluate the data. You decide to throw away all data points that have a $\perp$ on either detector and you calculate the correlations for the rest of them. You find that the data violates the Bell inequality.

d) Is it permissible to conclude with certainty that the local hidden variable theory proposed here is invalid? If not, what additional assumptions need to be made?

e) How do we have to do our post-processing if we do not want to make further assumptions? Remembering Tsirelson's inequality, give a lower limit on η, such that a violation of the Bell inequalities can still be detected.

The naive experimentalist
Now put yourself in the position of nature — or God, if you prefer: Your aim is to trick the experimentalist into believing that the local hidden variable model fails even though it does not.

*f) Devise a local strategy that produces data violating (2) if the experimentalist simply throws away $\perp$ events. How large will η get in your strategy if you want to reach Tsirelson's bound? The experimentalist is not stupid, he will get suspicious if your strategy only gives "no click"s on one detector or prefers some of the outputs.