Exercise 8.1 Tsirelson's Inequality

Tsirelson's inequality (cf. Nielsen/Chuang, Problem 2.3) gives an upper bound on the possible violation of Bell's inequality in Quantum mechanics. Let $Q = \vec{q} \cdot \vec{X},\ R = \vec{r} \cdot \vec{X},\ S = \vec{s} \cdot \vec{X}$ and $T = \vec{t} \cdot \vec{X}$ be observables with $|\vec{q}| = |\vec{r}| = |\vec{s}| = |\vec{t}| = 1$ and the Pauli matrices $\vec{X}$.

a) Show that all two-dimensional observables Q with eigenvalues ± 1 can be written in the form $Q = \vec{q} \cdot \vec{X}$.

b) Show that
$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4\mathbb{1} + [Q, R] \otimes [S, T]. \tag{1}$$

c) Argue that the expectation value can be bounded as $\langle \sqrt{X} \rangle \leq \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the maximum eigenvalue of a positive X.

d) Use the above results to prove Tsirelson's inequality:
$$\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle \leq 2\sqrt{2}. \tag{2}$$

e) We can generalize this to higher dimensions: Show that equation (1) holds for any set of observables with eigenvalues ± 1.

Exercise 8.2 Geometry of Measurements

In this exercise we will learn that the set of 2-outcome POVMs is a convex set with orthogonal measurements as extremal points.
Let $F = \{F_1, F_2\}$ and $G = \{G_1, G_2\}$ be two POVMs. We define an element-wise convex combination of F and G as $\alpha F + (1 - \alpha)G := \{\alpha F_1 + (1 - \alpha)G_1, \alpha F_2 + (1 - \alpha)G_2\}$, with $0 \leq \alpha \leq 1$.

a) Consider a POVM with two outcomes and respective measurement operators E and $\mathbb{1} - E$. Suppose that E has an eigenvalue λ such that $0 < \lambda < 1$. Show that the POVM is not extremal by expressing it as a nontrivial convex combination of two POVMs.
Hint: Consider the spectral decomposition of E and rewrite it as a convex combination of two POVM elements.

b) Suppose that E is an orthogonal projector. Show that the POVM cannot be expressed as a nontrivial convex combination of POVMs.

c) What is the operational interpretation of an element-wise convex combination of POVMs?