Quantum Field Theory II, Exercise Set ♯ 7.

FS 08/09 *Due: 29/4/2009*

1. Facts about Lie Algebras

a) Using eqs (5.17) and (5.18) in the Script, prove that

$$f_C^{AB} = -\frac{1}{2} Tr_{ad}\left([T^A, T^B]\, T^C\right) .$$

b) Starting from the relation (5.49) between the field strength tensor $F_{\mu\nu}$ and the covariant derivative $\nabla_\mu = \partial_\mu - A_\mu$, $F_{\mu\nu} = -[\nabla_\mu, \nabla_\nu]$, prove eq (5.44), namely $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$.

c) Starting from the Jacobi identity

$$[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] = 0\,,$$

verify eq (5.53),

$$\nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0\,.$$

2. Feynmann rules for non Abelian gauge theory

Consider a non-Abelian theory containing gauge bosons A_μ^A transforming under the fundamental representation of $SU(N)$ (we have QCD gluons in the back of our mind). We know that the corresponding Lagrangian is

$$\mathcal{L}_{kin} = -\frac{1}{4} F_{\mu\nu}^A F^{\mu\nu,A}\,,$$

with $F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_{\mu^A} + g f^{ABC} A_\mu^B A_\nu^C$.

In the lectures you have seen that the gauge-fixing term in the Lorentz gauge, $\partial_\mu A^{\mu,A} = 0$, is

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha}(\partial_\mu A^{\mu,A})^2\,,$$

and you wrote the Faddeev-Popov determinant as a Berezin integral over Grassmann ghost fields $\xi, \bar{\xi}$ with Lagrangian

$$\mathcal{L}_{ghosts} = \bar{\xi}_A \partial_\mu \left(\nabla_\mu^{AB} \xi_B\right) .$$

Here ∇_μ^{AB} is the covariant derivative in the adjoint representation,

$$\nabla_\mu^{AB} = \delta^{AB}\partial_\mu - g f^{ABC} A_\mu^C\,.$$

The total action is therefore given by

$$S = \int d^4x(\mathcal{L}_{kin} + \mathcal{L}_{gf} + \mathcal{L}_{ghosts})\,.$$

a) Write out in full the total action and carry out the various products therein so as to clearly see free terms and interaction terms.

b) Here we shall work towards the propagators.

   i) Let us focus first on the gauge field propagator.
   Write the part of the action containing only the kinetic terms for the A_μ^A fields. Be careful to include in particular the contribution from the gauge-fixing part. Then cast the integrand into a form such that derivatives only act on one of the gauge fields: integrate by parts and drop boundary terms, i.e., total derivatives in the integrand, and pass to momentum space. Recall that $f(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) \exp(ikx)$ and $\int d^4x \exp(ipx) = (2\pi)^4 \delta^{(4)}(p)$.
   Similarly to eq (5.91) in the Script, you should find that the propagator $D_{\nu\rho}$ for the gauge field is given by the condition

$$\left[g^{\mu\nu} k^2 - \left(1 - \frac{1}{\alpha}\right) k^\mu k^\nu \right] D_{\nu\rho}(k) = \delta^\mu_\rho \,.$$

   Since the only objects you can construct $D_{\nu\rho}$ from are the momenta k_μ and the metric tensor $g_{\mu\nu}$, the most general form for $D_{\nu\rho}$ is

$$D_{\nu\rho}(k) = A \frac{k_\nu k_\rho}{k^2} + B g_{\nu\rho} \,.$$

   Determine the coefficients A, B by substituting this into the LHS of the previous equation and requiring the equality to hold.

   ii) Repeat such calculations for the ghost propagator.

b) Let us now focus on the vertices.

   i) Write down the part of the action for the gluon-ghost-ghost interaction. Here, too, you can integrate by parts (and drop total derivatives), so that the derivative in these terms only acts on one field. Then write each field again in its momentum space representation and read off the vertex.

   ii) Write down the part of the action for the three gluons interaction. If it still looks like containing the difference of two terms, use the antisymmetry properties of the structure constant to prove that the second term is actually the opposite of the first one, so that they add up. Then write this term in a fully symmetric form. Again, exploit the cyclicity and antisymmetry properties of the structure constants and relabel the dummy indices where necessay, such that, in each term, the gauge field appears with the same indices. After passing to momentum space, read off the three-gluon vertex.

   iii) * Repeat such calculations for the four-gluon vertex.

c) Let us add a fermion field transforming under the fundamental representation of $SU(N)$,

$$\mathcal{L}_f = \bar{\psi} \left(i \not{\nabla} - m \right) \psi \,,$$

with $\nabla_\mu = \partial_\mu - igA_\mu^A T^A$. Determine the gluon-fermion-fermion vertex.

3. Axial gauge and Wilson line operators

Let us show that, given an arbitraty gauge field $A_\mu(x)$, it is always possible to find a gauge transformation g so that

$$^{g}A_3 = (gAg^{-1} + g\partial g^{-1})_3 = 0\,. \tag{1}$$

This can be done by considering the Wilson line operator

$$U(\gamma) = \int_\gamma \exp(dx^\mu A_\mu) = g$$

along a path γ that starts at a generic point X and runs parallel to the x_3 axis till the $x_3 = 0$ plane.
Show that along such path

$$\partial_3 g^{-1} = -A_3 g^{-1}\,,$$

and use it to prove eq (1).