Quantum Field Theory II, Exercise Set 6.

FS 08/09 *Due: 08.04.09*

1. Completing the Lecture Notes - I

Prove eq. (4.71) of the Lecture Notes, i.e.,

$$\int e^{-\xi^T A\xi + \mathrm{i}\eta^T\xi} d\xi_1 \cdots d\xi_n = e^{-\frac{1}{4}\eta^T A^{-1}\eta} 2^{n/2}\sqrt{\det A}\,, \tag{1}$$

A being an antisymmetrc $n \times n$ matrix, with n even.

Hint: Separate the term $i\eta^T\xi$ into $(i\eta^T\xi/2 + i\xi^T\eta/2)$, then find the fields ξ_{cl}, ξ_{cl}^T that minimize the exponent and use them to reshift the integration variables in such a way as to cancel the spurious terms $(i\eta^T\xi/2 + i\xi^T\eta/2)$. You should then be left with the evaluation of an integral of the form of eq (4.70) in the script. Work out a few examples ($n = 2$, $n = 4$) to convince yourself of the result - keep in mind that A is antisymmetric!

2. Completing the Lecture Notes - II

Let us prove eq. (4.6) from the Lecture Notes, i.e.,

$$\int e^{-|z|^2}|z\rangle\langle z|\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi\mathrm{i}} = 1\,. \tag{2}$$

a) Show that

$$\frac{1}{2\pi\mathrm{i}}\int e^{-a\overline{z}z}\mathrm{d}\overline{z}\wedge \mathrm{d}z = \frac{1}{a}\,. \tag{3}$$

Hint: Write the complex number z as $z = x + \mathrm{i}y$, and carefully work out the Jacobian. Then the integral reduces to the usual Gaussian one!

b) Argue that integrals of the form

$$\int z^n\overline{z}^m e^{-|z|^2}|z\rangle\langle z|\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi\mathrm{i}} \tag{4}$$

vanish unless $n = m$. You can simply focus on the case with $m = 0$. Then compute the integral

$$\int (z\overline{z})^n e^{-|z|^2}|z\rangle\langle z|\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi\mathrm{i}} \tag{5}$$

by noticing that it can be obtained from (3) by taking derivatives with respect to a.

c) Multiply both sides of equation (1) by $\langle\psi|$ on the left and by $|\chi\rangle$ on the right. Using the relations (4.3) - (4.8) in the Script and the result from point b), prove that the LHS reduces indeed to $\langle\psi|\chi\rangle$.

3. QED: Wick's theorem and the path integral formulation

The generating functional for the Dirac field is

$$Z(\overline{\eta},\eta) = \mathcal{N}\int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left[\mathrm{i}\int d4x\,\{\overline{\psi}(x)(\mathrm{i}\partial\!\!\!/ - m)\psi(x) + \overline{\eta}\psi + \overline{\psi}\eta\}\right]\,, \tag{6}$$

with the normalization factor $\mathcal{N}$ chosen so that $Z(0,0)=1$. Form this equation it is clear that for example the 2-points function is given by

$$\langle 0|T\psi(x_1)\overline{\psi}(x_2)|0\rangle = \mathcal{N}^{-1}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta\overline{\eta}(x_1)}\right)\left(-\frac{1}{\mathrm{i}}\frac{\delta}{\delta\eta(x_2)}\right)Z(\overline{\eta},\eta)\Bigg|_{\overline{\eta},\eta=0}\,, \tag{7}$$

with $\eta,\overline{\eta}$ anticommuting generators. On the other hand, using eq (4.68) of the Script, (6) can be rewritten as

$$Z(\overline{\eta},\eta) = \mathcal{N}\exp\left[-\int \mathrm{d}^4x\mathrm{d}^4y\,\overline{\eta}(x)S_F(x-y)\eta(y)\right]\,, \tag{8}$$

where the fermionic propagator $S_F(x-y)$ is defined via

$$\mathrm{i}(\mathrm{i}\partial\!\!\!/ - m)S_F(x-y) = -\delta^{(4)}(x-y)\,. \tag{9}$$

a) Using eq (8) and the RHS of eq (7), prove that indeed

$$\langle 0|T\psi(x_1)\overline{\psi}(x_2)|0\rangle = S_F(x-y)\,. \tag{10}$$

b) Proceeding in a similar way, compute $\langle 0|T\psi(x_1)\overline{\psi}(x_2)\psi(x_3)\overline{\psi}(x_4)|0\rangle$ and compare your result with what you would obtain by a straightforward application of Wick's theorem.

4. Finite Temperature Field Theory

In this exercise we apply the path integral formalism to finite temperature field theory. We consider a scalar field theory with Hamiltonian $H := \int \mathrm{d}^3k\varepsilon(\mathbf{k})a^*(\mathbf{k})a(\mathbf{k})$, where, e.g., $\varepsilon(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$. The grand-canonical partition function at inverse temperature β is defined by

$$Z_\beta := \mathrm{Tr}(\mathrm{e}^{-\beta(H-\mu N)})\,, \tag{11}$$

where μ is the chemical potential and $N := \int \mathrm{d}^3ka^*(\mathbf{k})a(\mathbf{k})$.

a) Show that the grand-canonical partition function can be written as

$$Z_\beta = \int \mathcal{D}\overline{\alpha}\wedge\mathcal{D}\alpha\,\langle\alpha|(\mathrm{e}^{-\beta(H-\mu N)})|\alpha\rangle\mathrm{e}^{-\int\mathrm{d}^3k|\alpha(\mathbf{k})|^2}\,, \tag{12}$$

where $|\alpha\rangle = \mathrm{e}^{\int\mathrm{d}^3k\alpha(\mathbf{k})a^*(\mathbf{k})}|0\rangle$.

Hint: Insert $\mathbb{1} = \int\mathcal{D}\overline{\alpha}\wedge\mathcal{D}\alpha\,|\alpha\rangle\langle\alpha|\mathrm{e}^{-\int\mathrm{d}^3k|\alpha(\mathbf{k})|^2}$ *in the definition of the trace.*

b) Using equation (4.25) in the lecture notes, show that grand-canonical partition function is given by

$$Z_\beta = \int \mathcal{D}\overline{\alpha}\wedge\mathcal{D}\alpha\,\mathrm{e}^{-\int_0^\beta\mathrm{d}s\int\mathrm{d}^3k\overline{\alpha}(\mathbf{k},s)(\frac{\partial}{\partial s}-\mu+\varepsilon(\mathbf{k}))\alpha(\mathbf{k},s)}\,, \tag{13}$$

with boundary conditions $\alpha(\mathbf{k},\beta)=\alpha(\mathbf{k},0)$ and $\overline{\alpha}(\mathbf{k},\beta)=\overline{\alpha}(\mathbf{k},0)$.

The thermal average of an operator $\mathscr{O}$ at inverse temperature β is given by

$$\left\langle\mathscr{O}\right\rangle_\beta = \frac{1}{Z_\beta}\mathrm{Tr}(\mathscr{O}\mathrm{e}^{-\beta(H-\mu N)})\,. \tag{14}$$

It can be expressed in terms of the temperature ordered Green's functions are defined by

$$\begin{aligned} &G^{(2n)}(\mathbf{k}_1, s_1; \dots \mathbf{k}_n, s_n \,|\, \mathbf{k}_{n+1}, s_{n+1} \dots \mathbf{k}_{2n}, s_{2n}) \\ &\qquad := \frac{1}{Z_\beta} \mathrm{Tr}\big(\mathrm{e}^{-\beta(H-\mu N)} \mathrm{T}\left[a(\mathbf{k}_1, s_1) \dots a(\mathbf{k}_n, s_n) a^*(\mathbf{k}_{n+1}, s_{n+1}) \dots a^*(\mathbf{k}_{2n}, s_{2n})\right]\big) , \end{aligned} \tag{15}$$

where $a(\mathbf{k}, t) = \mathrm{e}^{tH} a(\mathbf{k}, 0)\mathrm{e}^{-tH}$ and $a^*(\mathbf{k}, t) = \mathrm{e}^{tH} a^*(\mathbf{k}, 0)\mathrm{e}^{-tH}$, $0 \le t \le \beta$.

We define a generating functional $Z_\beta[J, \overline{J}]$ by

$$\begin{aligned} Z_\beta[J, \overline{J}] := \int \mathcal{D}\overline{\alpha} \wedge \mathcal{D}\alpha \, \exp\Big[-\int_0^\beta \mathrm{d}s \int \mathrm{d}^3k \big\{ &\overline{\alpha}(\mathbf{k}, s)(\frac{\partial}{\partial s} - \mu + \varepsilon(\mathbf{k}))\alpha(k, s) \\ &+ \overline{\alpha}(\mathbf{k}, s)\overline{J}(\mathbf{k}, s) + \alpha(\mathbf{k}, s)J(\mathbf{k}, s) \big\} \Big] , \end{aligned} \tag{16}$$

where we impose the same periodic boundary conditions as in (13).

c) Find an expression for the temperature ordered Green's functions in terms of the generating functional $Z_\beta[J, \overline{J}]$.

d) Show that

$$Z_\beta[J, \overline{J}] = \exp\left[-\int_0^\beta \mathrm{d}s \int \mathrm{d}^3k \, J(\mathbf{k}, s) \frac{1}{\frac{\partial}{\partial s} - \mu + \varepsilon(\mathbf{k})} \overline{J}(\mathbf{k}, s) \right] . \tag{17}$$

e) Expanding J and $\overline{J}$ in Fourier series, show that

$$Z_\beta[J, \overline{J}] = \exp\left[-\int_0^\beta \mathrm{d}s \int_0^\beta \mathrm{d}s' \int \mathrm{d}^3k \, J(\mathbf{k}, s)\overline{J}(\mathbf{k}, s')G(\mathbf{k}, s - s') \right] , \tag{18}$$

where

$$G(\mathbf{k}, t) = \frac{1}{\beta} \sum_{\omega_n \in \frac{2\pi}{\beta}\mathbb{Z}} \frac{\mathrm{e}^{-\mathrm{i}\omega_n t}}{-\mathrm{i}\omega_n - \mu + \varepsilon(\mathbf{k})} . \tag{19}$$

Hint: $J(\mathbf{k}, s) = \frac{1}{\beta} \sum_{\omega_n \in \frac{2\pi}{\beta}\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega_n s} \hat{J}(k, \omega_n)$, $\hat{J}(\mathbf{k}, \omega_n) = \int_0^\beta \mathrm{d}s \mathrm{e}^{-\mathrm{i}\omega_n s} J(\mathbf{k}, s)$.

f) Carrying out the sum in (19) explicitely show that

$$G(\mathbf{k}, t) = \mathrm{e}^{-t(\varepsilon(\mathbf{k})-\mu)} \left(\frac{1}{1 - \mathrm{e}^{-\beta(\varepsilon(\mathbf{k})-\mu)}} \theta(t) + \frac{\mathrm{e}^{-\beta(\varepsilon(\mathbf{k})-\mu)}}{1 - \mathrm{e}^{-\beta(\varepsilon(\mathbf{k})-\mu)}} \theta(-t) \right) . \tag{20}$$

Hint: The denominator may be written as $\frac{1}{a} = \int_0^\infty \mathrm{d}\lambda \mathrm{e}^{-\lambda a}$. *The summation over* ω_n *can be carried out using the Poisson summation formula, i.e., use* $\sum_{n=-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} n \frac{x}{\beta}} = \beta \sum_{k=-\infty}^{\infty} \delta(x - k\beta)$. *Finally, treat the cases* $t > 0$ *and* $t < 0$ *seperately and pay attention to the fact that* $\frac{|t|}{\beta} < 1$.