Quantum Field Theory II, Exercise Set 4.

FS 08/09 *Due: 24.03.09*

1. Convex functions and Legendre transform

First we recall what it means for a function to be convex: A function $f : D \to \mathbb{R}$ ($D \subset \mathbb{R}^n$) is convex, if its epigraph is a closed convex set in $\mathbb{R}^{n+1}$. The ephigraph is defined as

$$\Gamma(f) := \{(x,\, t) \,:\, x \in D,\, t \geq f(x)\}\,. \tag{1}$$

This definition implies, of course, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\,,\ 0 \leq \lambda \leq 1$. One observes that convexity implies continuity of f in $\mathrm{Int}(D)$. The *Legendre* transform f^* of f is defined for those $p \in \mathbb{R}^n$ for which

$$f^*(p) = \sup_{x \in D} \left((p,\, x) - f(x)\right) \tag{2}$$

is finite. D^* is then the set of such p. Show that

a) $f^* : D^* \to \mathbb{R}$ is convex.

b)* if f is convex, then $(f^*)^* = f$

c) if $f = f(x,\, y)$ is convex in $(x,\, y)$ then $f^*(p,\, y) = \sup_x[(p,\, x) - f(x,\, y)]$ is concave in y.

We restrict our discussion to the case where $n = 1$ and f convex.

d) Let p_0 be the slope of a supporting tangent line at x_0, i.e., $f(x) \geq f(x_0) + p_0(x - x_0)$. Find $f^*(p_0)$. Note that if f is differentiable then $p_0 = f'(x_0)$. Moreover, if f is strictly convex we have $x_0 = (f')^{-1}(p_0)$.

e) When can the supremum in (2) be dropped? Note that this has been used in the lecture notes for diverse explicite calculations. Finally, convince yourself that if $f(x)$ contains a linear piece from x_1 to x_2 with slope p_0, then $f^*(p_0)$ has a cusp, i.e., it is continuous, but not differentiable. However, the left- and right-derivative at $f^*(p_0)$ exist and are equal to x_1 and x_2, respectively.

2. Effective potential of the anharmonic oscillator

The aim of this exercise is to determine the effective potential of the anharmonic oscillator to one-loop order. Its Hamiltonian is

$$H := \frac{1}{2}P^2 + \frac{\mu^2}{2}Q^2 + \frac{\lambda}{4!}Q^4\,. \tag{3}$$

Let $H_J := H - JQ$, where J is a source term to be chosen appropriately in the following. The euclidean action corresponding to H is given by

$$S(q) = \frac{1}{\lambda}\int \mathrm{d}s \left(\frac{1}{2}\,\dot{q}(s)^2 + U(q(s))\right)\,, \tag{4}$$

where the potential is given by

$$U(q) = \frac{\mu^2}{2} q^2 + \frac{1}{4!} q^4 \,. \tag{5}$$

Note that we have rescaled the variables $q \to \sqrt{\lambda} q$. As in the lecture notes we choose a source term $J(t) = J_0 \chi_{[0,T]}$ such that $\frac{\delta W(J)}{\delta J} = \langle \Omega_{J_0} | Q | \Omega_{J_0} \rangle =: q_c$ is constant. We set $q(s) =: q_c + \xi(s)$.

The one-loop expansion is obtained by expanding the potential $U(q)$ up to second order in ξ around q_c.

a) Determine the effective action $\Gamma(q_c)$ to one-loop order.

Hint: Choose $J_0 = U'(q_c)$ and use 1e).

The effective potential is defined as

$$V(q_c) := \lim_{T\to\infty} \frac{1}{T} \Gamma(q_c) \,. \tag{6}$$

b) Show that the effective potential is given to one loop order by

$$V(q_c) = \frac{1}{\lambda} \left(\frac{\mu^2}{2} q_c^2 + \frac{1}{4!} q_c^4 \right) + \frac{1}{2} \int \frac{\mathrm{d}k}{2\pi} \log(k^2 + \mu^2 + \frac{1}{2} q_c^2) \,. \tag{7}$$

Remark: More precisely, the effective potential is given by the convex envelope of (7).

3. Effective potential and renormalisation

The one-loop contribution to the effective potential in dimension d is given by

$$V_{\mathrm{1Loop}}(\phi_c) = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log(k^2 + U''(\phi_c)) \,, \tag{8}$$

where the momentum integration is cut-off at some Λ. See equation (3.85) in the lecture notes.

a) Compute this integral in dimension one and four.

b) Fix the counter terms in dimension one and four using eq (3.87) in the lecture notes.

c)* Repeat the above steps for general dimensions.