Quantum Field Theory II, Exercise Set ♯ 1.

FS 08/09 *Due: 04.03.09*

1. Towards the SU(3) flavour group

In the lecture it was shown that the symmetry group G = SU(2) × SU(2) is not enough to explain the spectrum of observed baryons and mesons. For this reason we added to the set of relations

$$[H^i, E^i_\pm] = \pm 2 E^i_\pm \quad , \quad [E^i_+, E^i_-] = H^i \quad , \quad [H^1, H^2] = 0$$

The requirement that if $|v\rangle$ is simultaneously an eigenvector of H^1 and H^2 with eigenvalues m_1, m_2, respectively, then $E^j_\pm|v\rangle$ is again an eigenvector of H^1 and H^2.

a) From the definitions of the H^i, explain why the eigenvalues have to be integers.

b) Show that our previous requirement can be translated into

$$[H^i, E^j_\pm] = \pm A^{ij} E^j_\pm \,,$$

with $A^{ij} \in \mathbb{Z}$.

c) Show that the operators $E^\theta_\pm := [E^1_\pm, E^2_\pm]$ and $[E^1_\pm, E^2_\mp]$ are raising and lowering operators for (H^1, H^2).

Hint: A is a raising operator for N if $[N, A] = \alpha A$. In fact, if we denote by $|n\rangle$ an eigenvector of N and by n its eigenvalue, we immediately obtain

$$N(A|n\rangle) = (\alpha A + AN)|n\rangle = \alpha A|n\rangle + An|n\rangle = (n + \alpha)(A|n\rangle)\,.$$

d) Starting from $[E^i_\pm, [E^i_\mp, E^j_\pm]]$ and using the Jacobi identity, show that if we require $A^{ij} \neq 0$, then at least one set of raising/lowering operators from point c) does not vanish. Without loss of generality, we can assume these to be E^θ_+ and E^θ_-.

e) Let us impose the further conditions

$$[E^1_\pm, E^2_\mp] = 0 \quad \text{and} \quad [E^i_\pm, [E^1_\pm, E^2_\pm]] = 0\,.$$

Show that $A^{12} = A^{21} = -1$.

2. SU(3) representations

a) Starting from

$$(n, 0) \otimes (0, m) = (n, m) \oplus [(n-1, 0) \otimes (0, m-1)]$$

prove that

$$\dim(n, m) = \frac{1}{2}(n+1)(m+1)(n+m+2)\,.$$

b) Mesons are formed by a quark ($\mathbf{3} = (1, 0)$) and an antiquark ($\bar{\mathbf{3}} = (0, 1)$). Argue that

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}\,. \tag{1}$$

c) Baryons are formed instead by three valence quarks. Argue that

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}\,.$$

d) The strange, up and down quarks transform according to the **3** representation of SU(3). This three dimensional representation contains an isospin singlet (the strange quark) and an isodoublet. In addition, without loss of generality, we can assign hpyercharge y to the singlet and $y+1$ to the doublet. Therefore when we restrict the **3** of SU(3) to the isospin-hypercharge subgroup, we obtain

$$(1, 0)\Big|_{SU(2)_{\text{isospin}} \times U(1)_Y} = \left(\frac{1}{2}\right)^{\frac{1}{3}} \oplus (0)^{-\frac{2}{3}} \qquad , \tag{2}$$

where we denote by $(i)^y$ a state with isospin i and hypercharge y.

i) Show that $Y \in \mathrm{su}(3)$ indeed implies $y = -2/3$.

ii) Using the above decomposition, the Gell-Mann–Nishijima formula and $S = Y - B$ (the baryon number B being $1/3$ for the quarks), find the quantum numbers I^z, Y, Q, S for these three quarks.

e) Using (1) and (2), find the quantum numbers of the baryon octet.

Hint: Antiquarks can still be arranged into an isodoublet and an isosinglet, but their hypercharge changes sign. What is therefore the equivalent of (2) for $(0, 1)$*?*

3. Gauge field and covariant derivative

Let ρ be a unitary irreducible representation of a compact Lie group G on a vector space V. A field ϕ is a V-valued function over $\mathbb{M}^4$. A G-gauge transformation of ϕ is defined by

$$\phi(x) \mapsto \phi^g(x) = \rho(g(x))\phi(x)\,.$$

where $g(x) \in G\,, x \in \mathbb{M}^4$, is a G-valued function, assumed to be C^1. For definiteness we choose $G = \mathrm{SU}(N)$ and ρ is taken to be the fundamental representation. By definition, we have $V = \mathbb{C}^N$, and we might suppress ρ in the notation.

Consider the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi(x))^* \partial^\mu \phi(x) - m^2 \phi(x)^* \phi(x)\,.$$

a) Show that the given Lagrangian is invariant if g is a *global* transformation, i.e. $\partial_\mu g(x) = 0$.

b) How does $\mathcal{L}$ transform instead if $g = g(x)$, i.e., the transformation is *local*? Which term renders it not invariant?

Since we want the Lagrangian to be invariant, we introduce the notion of covariant derivative:

$$D^A_\mu := \partial_\mu - \mathrm{i}q\dot{\rho}(A_\mu)\,,$$

where A_μ is the non-abelian gauge field, which takes values in the Lie algebra $\mathrm{Lie}(G)$ of G. This fact will be shown later on in the lecture. Here $\dot{\rho}$ is the representation of $\mathrm{Lie}(G)$ induced by the representation ρ of G (see exercise set 1, QFT I), as above we suppress the $\dot{\rho}$ in our notation.

c) Find the transformed gauge field A^g_μ, if we require that the Lagrangian

$$\mathcal{L} = (D^A_\mu \phi(x))^* D^{A\mu} \phi(x) - m^2 \phi(x)^* \phi(x)\,.$$

be G-gauge invariant.

Hint: The Lagrangian is invariant if

$$(D^{A^g}_\mu \phi^g)(x) = g(x) D^A_\mu \phi(x)\,.$$

According to which representation of G transforms the covariant derivative?

For the construction of a gauge invariant kinematic term of the gauge field A_μ the covariant derivate is used itself.

e) How does the object $F_{\mu\nu} := -[D^A_\mu, D^A_\nu]$ transform under a gauge transformation? Which operation can we still perform on it to obtain a gauge invariant object?