Exercises for "Phenomenology of Particle Physics II"

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Exercise 9 Evolution of the coupling constant

The evolution of the strong coupling constant is given by

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) = -\alpha_s \left(\frac{\alpha_s}{2\pi}\beta_0 + \left(\frac{\alpha_s}{2\pi}\right)^2 \beta_1 + \cdots \right)$$

with the coefficients given by

$$\beta_0 = \frac{11C_A - 4T_R n_f}{6}$$
$$\beta_1 = \frac{16C_A^2 - (6C_F + 10C_A)T_R n_f}{6}.$$

$(C_F = \frac{N_C^2-1}{2N_C},\ C_A = N_C,\ T_R = \frac{1}{2})$

(i) Solve the evolution equation at one-loop ($\beta_0 \neq 0, \beta_1 = 0$) with the boundary condition $\alpha_s\left(\Lambda_{QCD}^{(0)}\right) = \infty$. Calculate $\Lambda_{QCD}^{(0)}$ from $\alpha_s(M_Z^2) = 0.118$, $n_f = 5$ and $M_Z = 91.187$ GeV.

(ii) Solve the evolution equation for $\beta_0 \neq 0$, $\beta_1 \neq 0$ with the boundary condition $\alpha_s\left(\Lambda_{QCD}^{(1)}\right) = \infty$ (only solve for α_s, do not invert the equation to get $\Lambda_{QCD}^{(1)}$).

Hints:

(i)
- Solve the differential equation by direct integration, keep track of the integration constants.
- Use $\alpha_s\left(\Lambda_{QCD}^2\right) = \infty$ to bring your result in the form

$$\alpha(\mu^2) = \frac{1}{k \log(\frac{\mu^2}{\Lambda_{QCD}^2})}$$

(ii)

- There is no analytic solution, consider an approximate solution by doing a series expansion of α_s in powers of $\log\left(\frac{\mu^2}{\Lambda_1^2}\right)$.
- Bring your result in the following form:

$$\alpha_s = \text{terms involving } \alpha_s \text{ inside log argument.}$$

- Solve the equation to first order in $\frac{1}{\log\left(\frac{\mu^2}{\Lambda_1^2}\right)}$. Insert this α_s on the right-hand side.
- Solve the resulting equation for large $\log\left(\frac{\mu^2}{\Lambda_1^2}\right)$, keeping terms up to $\left(\log\left(\frac{\mu^2}{\Lambda_1^2}\right)\right)^{-2}$. The following series expansion can be used:

$$\frac{1}{1 + c_1 x \log(c_3 \frac{1}{x} + c_2)} = 1 + x c_1 \left(\log(x) - \log(c_3)\right) + O(x^2).$$