Proseminar FS09 in Theoretical Physics - Perturbative and non-perturbative methods for strong interactions

The finite-temperature transition in QCD

Raffaele Solcà
ETH Zürich
Dr. Philippe de Forcrand
Tutor: Dr. Aleksi Kurkela

18th May, 2009

Contents

1 Introduction

One of the topics of modern research of QCD is the study of the properties of the transition that will take place if we heat sufficiently a system of strongly interacting particles.

We will see a simple example that can be analytically understood, the Bag model, and then we will see what types of transitions can take place in QCD at various quark masses.

2 Thermodynamics

Let us first introduce some concepts of thermodynamics, which are useful to understand what will be done in the next sections.

We define a system to be a grand canonical ensemble if it has a fixed temperature T, a chemical potential μ and a volume V. A physical construction of such a system can be done by taking a big reservoirs, for heat at a temperature T and for particles with a chemical potential μ. In a relativistic system the particles can be created and annihilated, hence we compute the observable in the grand canonical ensemble.

From statistical quantum mechanics it is known that a grand canonical ensemble with an Hamiltonian H and some conserved charges N_i which commute with H has the following density matrix

$$\rho = \exp\left(-\beta\left(H - \sum_i \mu_i N_i\right)\right). \tag{1}$$

It follows that the density operator is diagonal in the energy eigenstate $\{|n\rangle\}$. It is also known that the eigenvalues of this operator are the probabilities to find the system to be in the eigenstate $|n\rangle$. The partition function is then defined to be

$$Z = \mathrm{Tr}(\rho) = \sum_i \langle i\,|\rho|\,i\rangle \tag{2}$$

where $|i\rangle$ is a basis of the Hilbert space.

Now with the partition function we can represent all the thermodynamical functions such as the pressure P, the entropy S, the energy E , the particle number N_i, the free energy F, the grand potential Ω, etc. as follows

$$\begin{aligned} P &= \frac{\partial(T \ln Z)}{\partial V} \\ S &= \frac{\partial(T \ln Z)}{\partial T} \\ E &= TS - PV + N_i\mu_i \\ N_i &= \frac{\partial(T \ln Z)}{\partial \mu_i} \\ F &= -PV + N_i\mu_i \\ \Omega &= E - TS - N_i\mu_i = -PV \end{aligned}$$

2.1 Phase diagrams

Let us introduce this section with a well known example of a phase diagram. The diagram in Figure 1 is a sketch of the water phase diagram. The control parameters are in this case the temperature and the pressure, while the three different regions represent the three different phases of ice, water and steam. The lines mark a coexistence of two different phases. Further two special points are present in the diagram. The triple point where the three phases coexist and the critical point where the meniscus that separates the liquid and the gas phase disappear and the two phases are no more distinguishable. It is known that the phase transition between liquid and gas phases is a first order phase transition below the temperature of the critical point and is a second order phase transition in the critical point.

The diagram in Figure 2 is a proposed phase diagram for QCD. The names of the various phases are underlined in the picture, while the environment in which these can be found is not. Also in this diagram the lines show coexistence of two phases and the critical points are represented with filled points. The region in which there are crossover transitions are drawn in gray. The control parameters are in this case the temperature and the baryon number chemical potential. As we have seen in the previous section we use the grand canonical ensemble to describe a system in which particle can be created or destroyed. Hence the convenient potential to use to find the equilibrium is the grand potential $\Omega(T, V, \mu) = E - TS - B\mu$, where $B = N_B - N_{\bar{B}}$ is the baryon number.

First important remark about this diagram is that only a small region near the vertical axis is understood, because we can do lattice simulation only in this region. We can say that at the origin the universe was in the upper left corner, but after about hundred microseconds it was passed under

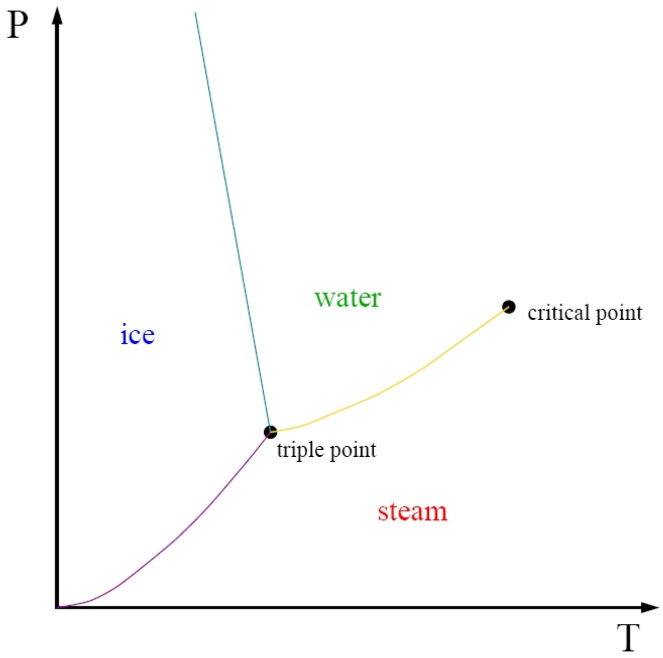

Figure 1: The phase diagram of water. (Taken from [3])

the critical temperature $T_c \approx 170$ Mev $\approx 2 \cdot 10^{12}$K. Nowadays the universe is in the lower left corner of this phase diagram.

Let us now introduce the last phase diagram that will be discussed in the next sections. In Figure 3 we see the phase diagram of the transition in QCD. The control parameters are in this case the mass of the lightest quark flavours up and down ($m_u = m_d$) and the mass of the strange quark m_s.

3 Bag model

In this section we use a simple model, called the Bag model, to analyze the phase transition between the gas of pions and the quark-gluon plasma (QGP).

The fundamental interaction between quarks in QCD is the exchange of gluons. Gluons are spin-1 particle and have a zero baryon number. They are like the photons for the electrical interaction between charges, but there is a big difference: the photons carry no electrical charge, whereas the gluons carry color charge which implies that gluons can also interact between them. Let us analyze how the potential between a quark and an antiquark behave with respect to the separation between them. In [12] was shown that

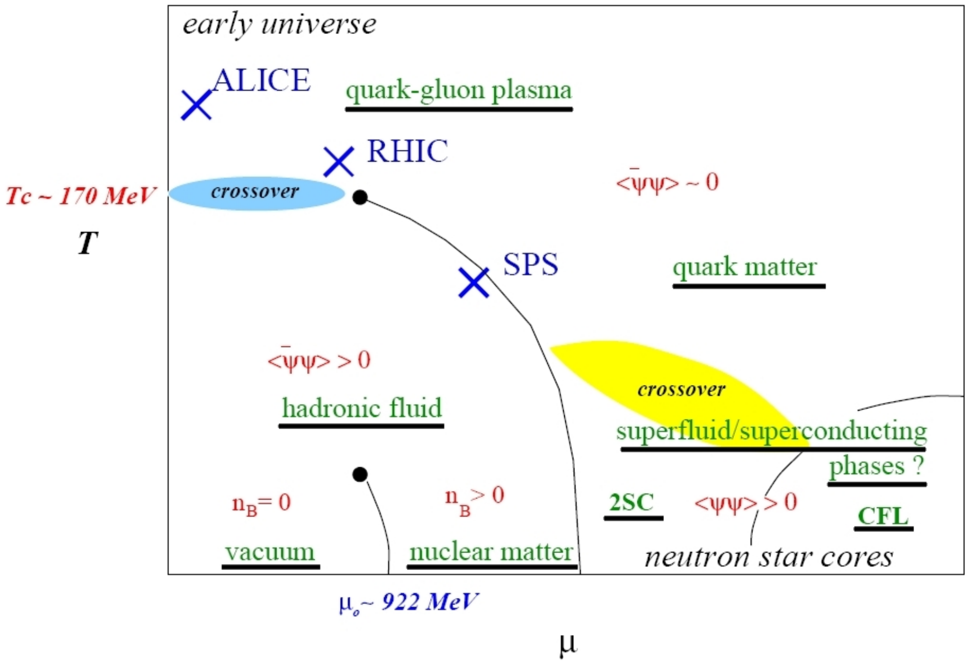

Figure 2: Proposed phase diagram for QCD. 2SC and CFL refer to di-quark condensate, SPS, RHIC and ALICE are the names of experiments with heavy-ions collision. This diagram is taken from [3]

for vanishing lattice spacing, i.e. for continuum limit, we have a vanishing coupling constant for small separation, that implies that interaction becomes arbitrarily weak for small separation. This fact is called asymptotic freedom and is confirmed by experiment.

In [12] was also considered the behaviour of the potential between quark and antiquark in the limit of big separation between the two particles in lattice theory. It was shown that the potential increase linearly with the separation, which implies that if we want to separate the two particles we will need an infinite amount of energy. This is true also for particles that are not in a color singlet state. This behaviour is called confinement, and it is also present in the continuum limit.

Let us introduce the Bag model, which is a simple model that take into account asymptotic freedom and confinement. We assume that the quarks can move freely inside the hadron (which is supposed to have a spherical shape) but they can not go away because there is a pressure that confines the quarks. This can be explained considering that there exists a true vacuum, which broke chiral symmetry, with a free energy smaller than the free energy

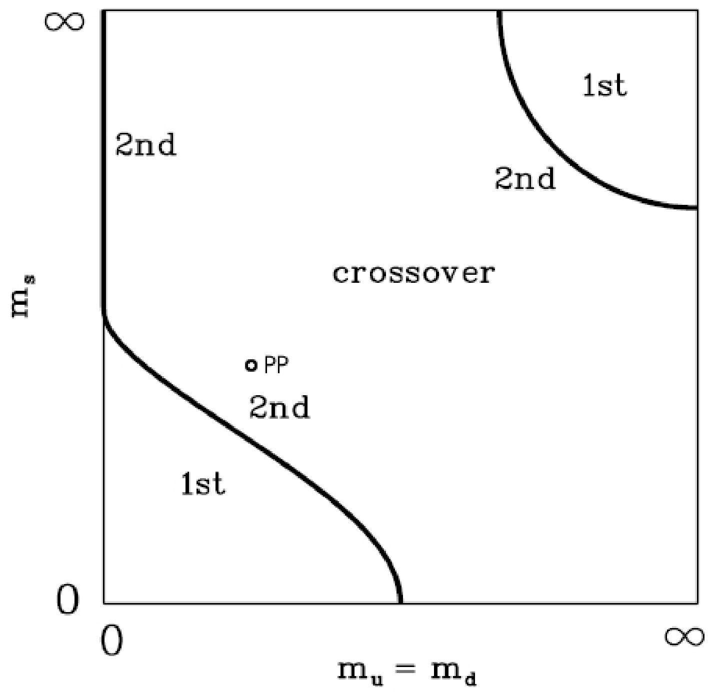

Figure 3: A possible phase diagram for QCD: Columbia Plot. This diagram shows the transition in function of the strange mass quark m_s and the up and down mass quarks ($m_u = m_d$). The physical point is denoted by PP. This diagram is taken from [1]

of the trivial vacuum. Let us assign the value of the free energy to the trivial vacuum and to the true vacuum.

$$F(\text{trivial vacuum}) = 0 \text{ and } F(\text{true vacuum}) = -\Lambda_B^4. \tag{3}$$

We assume that the hadrons replace R^3 true vacuum, where R is the radius of an hadron, with trivial vacuum. It follow that the free energy of the hadron is

$$F(\text{hadron}) = -\Lambda_B^4 V + \Lambda_B^4 (V - R^3) = \Lambda_B^4 R^3, \tag{4}$$

where the first term is the free energy of a system of volume V filled with true vacuum and the second term is the free energy of a system of volume V of true vacuum with a sphere of radius R of trivial vacuum. In addiction we have to consider the contribution of the Uncertainty Principle. It follows that the energy of an hadron is

$$E \approx R^3 \Lambda_B^4 + \frac{C}{R}, \tag{5}$$

where the last term is the kinetic energy due to the Uncertainty Principle. It follows that

$$M \approx 4R^3\Lambda_B^4. \tag{6}$$

Hence, using the values of a nucleon ($M \approx 1000$ MeV, $R \approx 1$ fm), we can find the value of the bag constant $\Lambda_B \approx 200$ MeV.

Let now consider strong interactions at a temperature $0 < T < T_c$. Hence we can consider only up and down quarks ignoring the other heavier quarks. We assume also that the rest mass of the up, down quarks and of the pions is ≈ 0, which is a good approximation for a temperature above 100 MeV, and we can assume that chiral symmetry holds. We assume also that there is no net concentration of baryons ($B = 0$). Hence the dominant degrees of freedom are pions, which are pseudo Goldstone boson (because the mass of the quarks is not exactly zero), The pions carry $B = 0$. Then the pressure due the pions gas can be calculated to be:

$$P_\pi = -\frac{\partial \Omega_\pi}{\partial V}\bigg|_{T,\mu} = 3 \times \left(\frac{\pi^2}{90}\right) T^4, \tag{7}$$

where we add a factor three counting the three different charge states of the pion. We can do this approximation because the pions weakly interact between themselves.

We now consider the system with a temperature $T > T_c$. Here the light quarks and gluons are no more confined in the hadrons, hence we have a QGP. Analogously to the pion gas the pressure becomes, due to the increased number of degrees of freedom,

$$P_{q\bar{q}} = 2 \times 2 \times 3 \times \frac{7}{4} \times \left(\frac{\pi^2}{90}\right) T^4 \quad ; \quad P_g = 2 \times 8 \times \left(\frac{\pi^2}{90}\right) T^4. \tag{8}$$

For the quark-antiquark pairs the numerical factors are: 2 for the two different helicity states, 2 because we consider two flavour of quarks, three for the different colors state, 7/4 comes from the difference between the Fermi-Dirac and the Bose-Einstein statistics. For the gluons the factor 2 comes from the two different helicity states and the factor 8 because the gluons can carry eight different states of color and anticolor-charge.

We now want to find the critical Temperature where the two phase coexists. Hence we assume that the two different states are in equilibrium if the pressure are coincident. However we have to consider the different pressure of the two different vacuum, hence we use the bag constant. We obtain that

$$\frac{1}{30}\pi^2 T_c^4 = \frac{37}{90}\pi^2 - \Lambda_B^4, \tag{9}$$

where we denote by T_c the critical temperature where the phase transition occurs. Calculating the critical temperature we obtain $T_c \approx 144$ MeV, which is equivalent to $T_c \approx 167 \cdot 10^{10}$ K.

4 QCD concepts

We will now look in details the phase transition for strongly interacting systems. We are interested in the order of the phase transition because if there is a first order phase transition the results of the experiments change. Strong interaction are described in QCD that use the concept of action. In [12] it was calculated the non-abelian Wilson action on a lattice for QCD. It looks like

$$\begin{aligned} S &= (\hat{m}+4r)\sum_n \bar{\psi}(n)\psi(n) \\ &\quad -\frac{1}{2}\sum_{n,\mu} \bar{\psi}(n)(r-\gamma_\mu)U_\mu(n)\psi(n+\mu) + \bar{\psi}(n+\mu)(r+\gamma_\mu)U_\mu^\dagger(n)\psi(n) \\ &\quad +\frac{2}{g^2}\,\mathrm{Tr}\sum_{n,\mu<\nu}\left[\mathbf{1}_3 - \frac{1}{2}(P_{\mu\nu}(n)+P_{\mu\nu}^\dagger(n))\right], \end{aligned} \tag{10}$$

where the first part is the action of the quark and the last part is the action of the gluons. $\psi(n)$ is the field of a fermion and $\bar{\psi}$ is that of an antifermion.

In finite temperature lattice theory the action on the lattice is periodic in time for bosons whereas it is antiperiodic for fermions.

5 The infinite quark mass limit

In this section we will consider the phase transition with infinite quark masses, i.e. we are considering the upper right corner of the Columbia Plot in Figure 3. If we assume that the quarks have an infinite mass, the quarks will not take part in the interactions. To analyze this situation we use pure gauge theory.

5.1 Center Symmetry

We use a lattice periodic in the time direction with no quarks in the sites and we take SU(3) as gauge group. The action (10) is defined to be invariant under gauge transformation.

If we apply at the links of one row as in Figure 5 a transformation Z, i.e. $U_\nu(\hat{m})$ becomes $ZU_\nu(\hat{m})$, where $\hat{m} = \hat{c} + k\hat{\mu}$, c fixed and $k \in \mathbb{Z}$, then

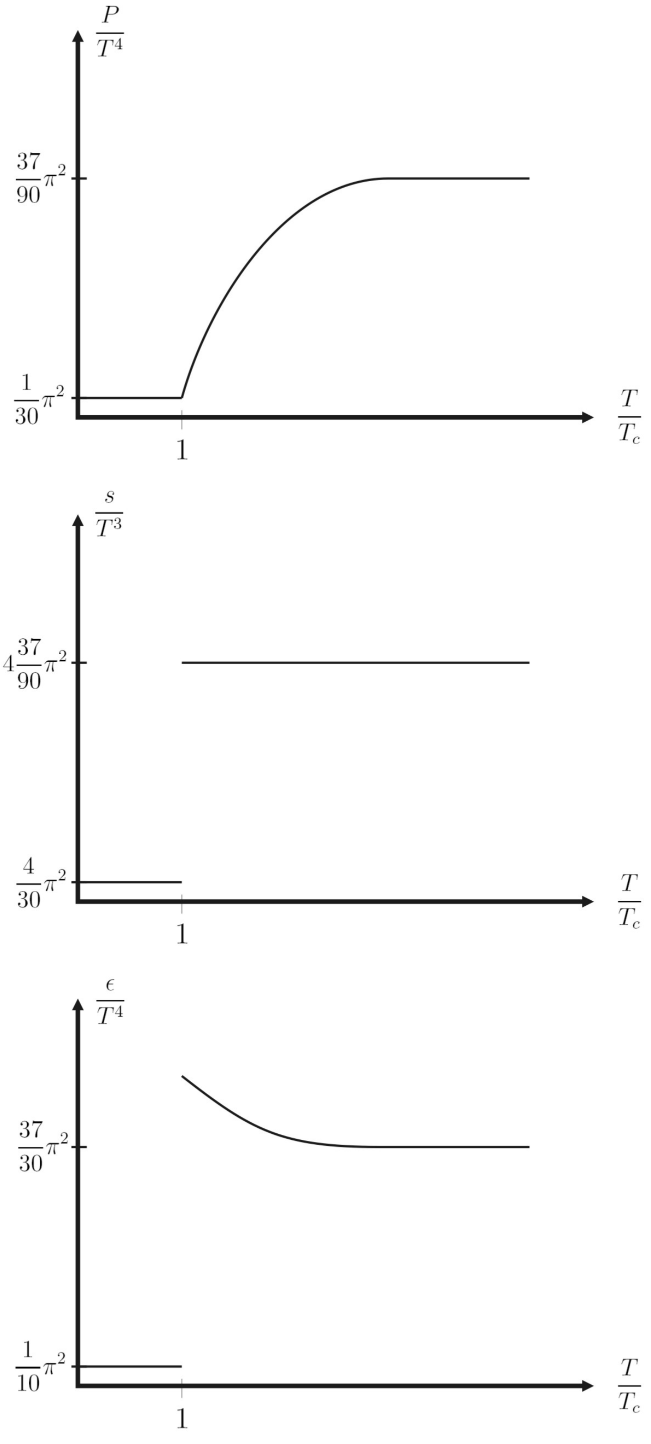

Figure 4: The equation of state of Bag model versus T/T_c.

calculating the transformation of the loop $P_{\mu\nu}(m)$ in the same figure we obtain

$$P_{\hat{\mu}\hat{\nu}} \to \left(U_{\hat{\mu}}(\hat{m}) Z U_{\hat{\mu}}(\hat{m}+\hat{\mu}) U^{\dagger}_{\hat{\mu}}(\hat{m}+\hat{\nu}) U^{\dagger}_{\hat{\mu}}(\hat{m}) Z^{\dagger} \right). \tag{11}$$

We impose that the action should be invariant under this transformation, so we need the transformation Z to commute with all the elements of the gauge group; the group of the transformations which satisfy this property is called the center of the group. For the SU(N) group the center is called Z(N) and is composed by the matrices $\{\exp(2i\pi n/N) \cdot \mathbf{1}\}$ with $n = 0, 1, \ldots n-1$ where with $\mathbf{1}$ we denote the identity matrix. Hence if we apply a transformation Z of the center of the group the action does not change: in this way we obtain a symmetry, called the center symmetry.

5.2 The Polyakov loop

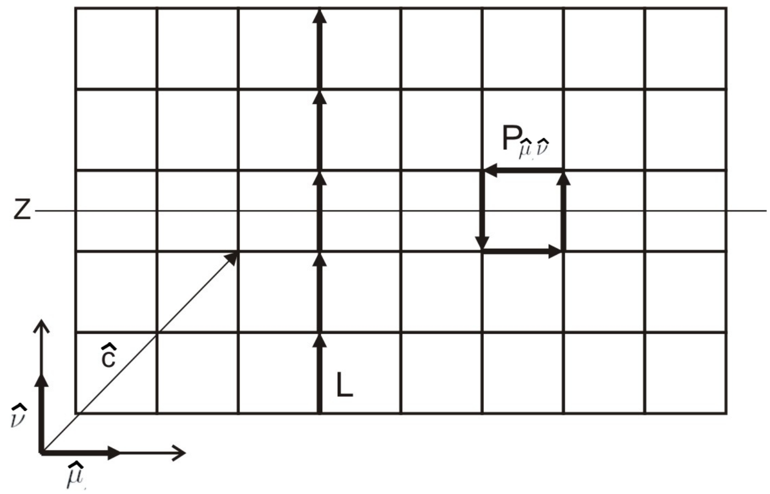

Figure 5: Example of a center transformation. The vertical axis represents the Euclidean time and the horizontal axis represents the space. All the plaquettes are invariant under a center transformation, while the Polyakov Loop L transform in the fundamental representation.

Let us consider the loop L, called Polyakov loop, as in Figure 5. Because of the periodicity of the lattice the Polyakov Loop is invariant under a gauge transformation. It is definite as:

$$L(\hat{m}) = \prod_{j=0}^{N_\tau - 1} U_{\hat{\nu}}(\hat{m} + j\hat{\nu}) \tag{12}$$

where N_τ is the number of sites of the lattice in temporal direction. Let us now see how this loop looks behave under the application of the transformation Z

$$L \to ZL. \tag{13}$$

In order to ensure the center symmetry we need the value of this quantity to be zero. If this value is different from zero, this is the manifestation that the center symmetry is spontaneously broken. Because of this fact we can use it as order parameter of pure gauge theory. Consider the function

$$L(T) = \begin{cases} 0, & T < T_c \\ L'(T) \neq 0, & T > T_c \end{cases}. \tag{14}$$

We have that an analytical function that is zero on an interval (a, b) with $a \neq b$ is always zero. Hence our function $L(T)$ is not analytical and further we are in presence of a phase transition. It is known that for Z(3) center symmetry there is a first order phase transition, If there would be only two different color charges, a second order phase transition would occur.

5.3 Center symmetry vs. fermions

We now allow the quarks to be in the lattice. If we now compute the action of the system we obtain that it is not invariant under application of transformation of the center of the gauge group because of

$$\begin{aligned} \bar{\psi}(\hat{m})(r - \gamma_{\hat{\mu}})U_{\hat{\nu}(\hat{m})}\psi(\hat{m} + \hat{\nu}) + \bar{\psi}(\hat{m} + \nu)(r + \gamma_{\hat{\nu}})U^\dagger_{\hat{\nu}}(\hat{m})\psi(\hat{m}) \quad &\to \\ \bar{\psi}(\hat{m})(r - \gamma_{\hat{\mu}})ZU_{\hat{\nu}}(\hat{m})\psi(\hat{m} + \hat{\nu}) + \bar{\psi}(\hat{m} + \nu)(r + \gamma_{\hat{\nu}})U^\dagger_{\hat{\nu}}(\hat{m})Z^\dagger\psi(\hat{m}). & \end{aligned} \tag{15}$$

Hence the center symmetry is broken if we add fermions in the lattice, and the Polyakov loop is no more an order parameter of the transition.

5.4 Computer simulations

Let us have a look at simulations of the expectation value of the Polyakov loop with respect to the temperature in the pure SU(3) gauge theory. In Figure 6, knowing that the expectation value of the Polyakov loop is zero below the critical temperature,we see that we are in presence of a phase transition.

Let us also have a look at other quantities with respect to the temperature. In Figure 7 we see the behaviour of the entropy, the energy density and of the pressure for the pure SU(3) gauge theory. We see also in this simulation that a first order phase transition occurs.

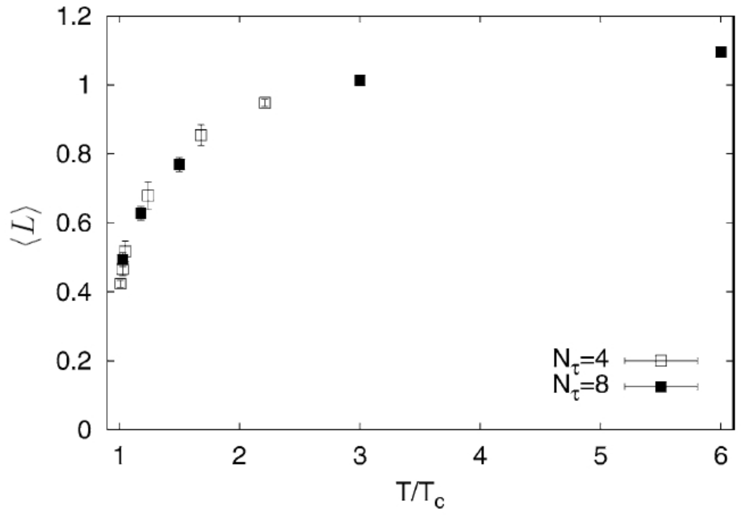

Figure 6: The expectation value of the Polyakov loop as a function of the temperature. The spatial lattice size in $N_s = 32^3$. The expectation value is zero below the critical temperature T_c. The graph and the data are taken from [5]

6 The massless quarks limit

6.1 Chiral Symmetry

As done in [11], let us define the chiral matrix γ_5 as follows

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{16}$$

that satisfy $\gamma^5 = \gamma_5^\dagger = \gamma_5$. Hence we can define the projection operators to the left- and right-handed.

$$P_L = (1 - \gamma^5) \text{ and } P_R = (1 + \gamma^5). \tag{17}$$

Let us introduce a Lagrangian with N_F flavours of massless fermions f that describes fields with independent left- and right-handed components.

$$\mathcal{L}^0 = \sum_f \left[\bar{q}_{f,L}(i\not{D})q_{f,L} + \bar{q}_{f,R}(i\not{D})q_{f,R}\right] - \frac{1}{4}\mathcal{G}_{a,\mu\nu}\mathcal{G}_a^{\mu\nu}, \tag{18}$$

where $\mathcal{G}^a_{\mu\nu}$ represents the gauge invariant gluonic field strength tensor. It can be easily verified that this Lagrangian is invariant under global phase shift

$$q_{f,J} \rightarrow e^{-i\theta_J} q_{f,J} \text{ where } J = R, L \tag{19}$$

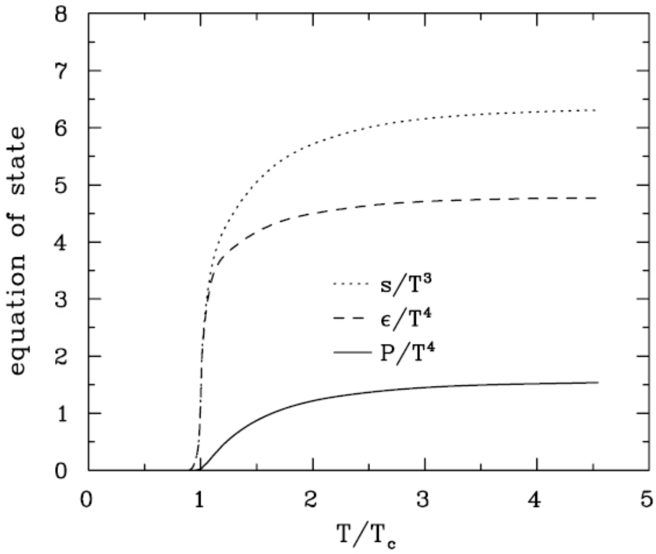

Figure 7: The equation of state of the pure SU(3) gauge theory versus T/T_c. This graph is taken from [1] and the data from [4]

which indicates that this Lagrangian has a $\mathrm{U}(1)_L\times$ $\mathrm{U}(1)_R$ symmetry.

We can also see that there is another symmetry by noting that the gauge invariant derivative is independent on flavour, hence rotating independently left- and right-handed components in flavour space as follows

$$\begin{pmatrix} f_{1R,L} \\ \vdots \\ f_{N_F R,L} \end{pmatrix} \rightarrow F_{R,L} \begin{pmatrix} f_{1R,L} \\ \vdots \\ f_{N_F R,L} \end{pmatrix}, \tag{20}$$

where $F_{L,R} \in \mathrm{SU}(N_F)$, will leave the Lagrangian invariant. Performing a transformation like the above one we find that the total symmetry group is $\mathrm{SU}(N_F)_R\times$ $\mathrm{SU}(N_F)_L\times$ $\mathrm{U}(1)_R\times$ $\mathrm{U}(1)_L$.

We can replace the $\mathrm{SU}(N_F)_R\times$ $\mathrm{SU}(N_F)_L$ symmetry with the axial and vector symmetry $\mathrm{SU}(N_F)_V\times$ $\mathrm{SU}(N_F)_A$. The transformation will be definite as

$$\text{Vec: } \begin{pmatrix} f_{1R,L} \\ \vdots \\ f_{N_F R,L} \end{pmatrix} \rightarrow e^{-i\theta_V} \begin{pmatrix} f_{1R,L} \\ \vdots \\ f_{N_F R,L} \end{pmatrix} \text{ and Ax: } \begin{pmatrix} f_{1R,L} \\ \vdots \\ f_{N_F R,L} \end{pmatrix} \rightarrow e^{-i\theta_A\gamma^5} \begin{pmatrix} f_{1R,L} \\ \vdots \\ f_{N_F R,L} \end{pmatrix} \tag{21}$$

6.2 Chiral condensate

Let us define the chiral condensate as

$$\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \rangle , \tag{22}$$

where ψ_L annihilates a left-handed quark and $\bar{\psi}_L$ creates a left-handed quark, for right handed quarks is analogous. Under a vectorial chiral transformation $F_V = e^{-i\theta_V}$ the chiral condensate will transform in the following way

$$\langle \bar{\psi}\psi \rangle \to \left\langle \bar{\psi}_L F_V^\dagger F_V \psi_R + \bar{\psi}_R F_V^\dagger F_V \psi_L \right\rangle = \langle \bar{\psi}\psi \rangle , \tag{23}$$

whereas under an axial transformation $e^{-i\theta_A \gamma^5}$

$$\langle \bar{\psi}\psi \rangle \to \langle \bar{\psi}_L e^{-i\theta_A} e^{-i\theta_A} \psi_R + \bar{\psi}_R e^{i\theta_A} e^{i\theta_A} \psi_L \rangle . \tag{24}$$

As we can see the chiral condensate is not invariant under an axial chiral transformation. It follows that in order to ensure the chiral symmetry we need that the value of the chiral condensate is zero. If this value will differ from zero, it imply that a spontaneous breaking of chiral symmetry takes place. Analogously to the value of the Polyakov loop we have a function that is zero in a interval and it differ from zero in another. Hence the function is not analytical and we are in presence of a phase transition.

We now know that there is a phase transition, but is first or second order phase transition? Let us assume that a second order phase transition occurs. It follow that the correlation length ξ diverges and because of that we can apply the universality argument. We have seen that only $\mathrm{SU}(N_F)_A$ is broken and hence it follow that $N_F^2 - 1$ Goldstone modes will appear in the broken symmetry phase. Let us take a theory with the same symmetry, i.e. $\mathrm{SU}(N_F)\times$ $\mathrm{SU}(N_F)$, in three dimension that have the same degrees of freedom. Let us introduce the system of $N_F \times N_F$ matrices Φ that transform in the following way

$$\Phi \to L\Phi R, \tag{25}$$

where L and R are two independent $\mathrm{SU}(N_F)$ matrices. Then we take the most general (super-) renormalizable Lagrangian invariant under symmetries, that is

$$\mathcal{L}_\Phi = c_o \mathrm{Tr}\left(\partial_\mu \Phi^\dagger \partial_\mu \Phi\right) - c_1 \mathrm{Tr}\left(\Phi^\dagger \Phi\right) - c_2 \left(\mathrm{Tr}\left(\Phi^\dagger \Phi\right)\right)^2 - c_3 \mathrm{Tr}\left(\Phi^\dagger \Phi\right)^2 - c_4 \mathrm{Re}\left(\det \Phi\right) . \tag{26}$$

Let now us assume that $\Phi = \lambda U$ where $U \in \mathrm{SU}(N_F)$ and $\lambda \in \mathbb{C}$, then the potential becomes

$$V = c_1 \left|\lambda\right|^2 + (c_2 + c_3) \left|\lambda\right|^4 + c_4 \mathrm{Re}\left(\lambda^{N_F}\right) . \tag{27}$$

If we consider the case with $N_F = 2$ we obtain a potential of the form

$$V = a\,|\lambda|^2 + b\mathrm{Re}\left(\lambda^2\right) + c\,|\lambda|^4\,, \tag{28}$$

that leads to a second phase transition. Oppositely if we consider the case with $N_F = 3$ we obtain a potential of the form

$$V = a\,|\lambda|^2 + b\mathrm{Re}\left(\lambda^3\right) + c\,|\lambda|^4\,, \tag{29}$$

which lead a first order phase transition or a crossover, because of the presence of terms of odd order. Hence for $N_F = 2$ we can say that there is a second order phase transition, whereas for N_F=3 there is no second order phase transition, but knowing that there is a phase transition, because the function that describe the chiral condensate is not analytic, we can say that there is a first order phase transition.

6.3 Chiral symmetry vs. massive quarks

If we put the mass term

$$\mathcal{L}^M = (\bar{q}_L + \bar{q}_R)M(q_L + q_R) \tag{30}$$

in the simplified Lagrangian (18) the resulting Lagrangian $\mathcal{L} = \mathcal{L}^0 - \mathcal{L}^M$ will no more be invariant under the transformation (19) (20), hence the chiral symmetry group $\mathrm{SU}(N_F)_R\times \mathrm{SU}(N_F)_L$ is broken. To see this fact take an axial transformation, for n flavours of quarks, that maps the quarks $f_R \to Af_R$ and $f_L \to A^\dagger f_L$ where f is the flavour of the quark and $A \in \mathrm{SU}(n)$. It follow that the mass term of the Lagrangian becomes

$$\mathcal{L}^M = \sum_f \left(\bar{f}_L + \bar{f}_R\right) m_f \left(f_L + f_R\right) \to \sum_f \left(\bar{f}_L A + \bar{f}_R A^\dagger\right) m_f \left(A^\dagger f_L + Af_R\right), \tag{31}$$

where f indicate the field of quarks of flavour f and $\bar{f}$ indicates the field of the f antiquarks. It easy to see that the mass term of the Lagrangian is not invariant under axial chiral transformations.

6.4 Computer simulations

Let us look at simulations with massless quarks. In Figure 8 we can see the behaviour of the expectation of the chiral condensate for three massless quarks. It should be a first order phase transition, but it is not seen. This fact can be explained considering that the massless limit is an extrapolation

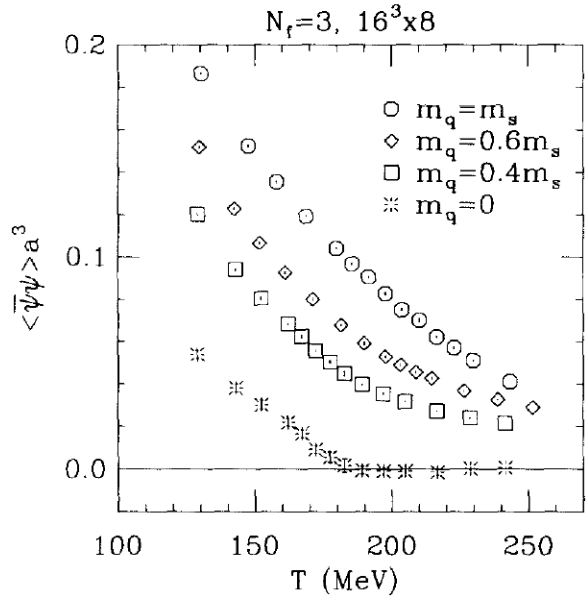

Figure 8: The chiral condensate $\langle\bar{\psi}\psi\rangle$ measured in lattice units in function of the temperature. The quarks mass are assumed to be the same for the up, down and strange flavour. The graph and data are taken from [8]

of the other lines with massive quarks. If the masses of the simulation are far from the massless limit the results does not show a first order phase transition.

Now fix the mass of the strange quark to its real mass and look at the results of the analogous simulation of the value of the chiral condensate in Figure 9. Analogously as in Figure5b the massless limit is an extrapolation and we cannot say what type of transition will take place. However we can say that below a temperature, we call the critical temperature the chiral condensate is different from zero (spontaneous chiral symmetry breaking), whereas above the critical temperature is zero.

In Figure 10 and 11 are shown the simulation of the pressure and the energy density in function of the temperature for three different cases. The first case is with two different flavours of light quarks, the second is with two flavours of light quarks and one of heavy quarks, while the last one is with three flavours of light quarks, where "light" means that $m = 0.4T$ and "heavy" means $m = T$.

Let us now see what will happen when we consider the physical point, i.e. the point with the real mass of the flavours of quark. In figure 12, 13, 14 and 15 simulation of the energy, pressure, entropy, chiral condensate and Polyakov loop are shown. We can see that the transition is a crossover transition. In Figure 16 we see that the critical temperature is almost the same for both transitions. This fact has not an explanation yet.

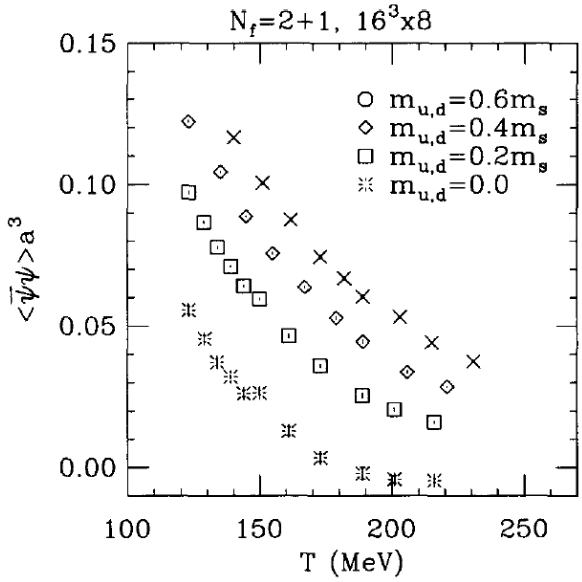

Figure 9: The chiral condensate $\langle\bar{\psi}\psi\rangle$ measured in lattice units in function of the temperature. The strange quark mass is fixed so that we have the physical value of the ϕ meson calculated on the lattice. The graph and data are taken from [8]

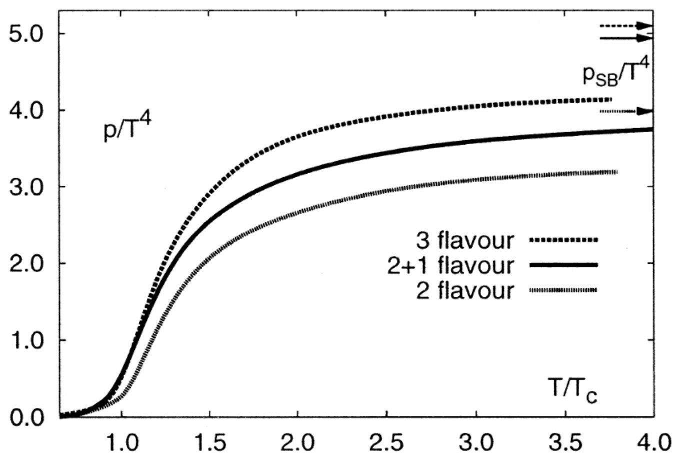

Figure 10: The pressure in units of T^4 versus T/T_c for two, three flavour of light quarks and for two flavour of light and one of heavy quarks. This graph is taken from [7]

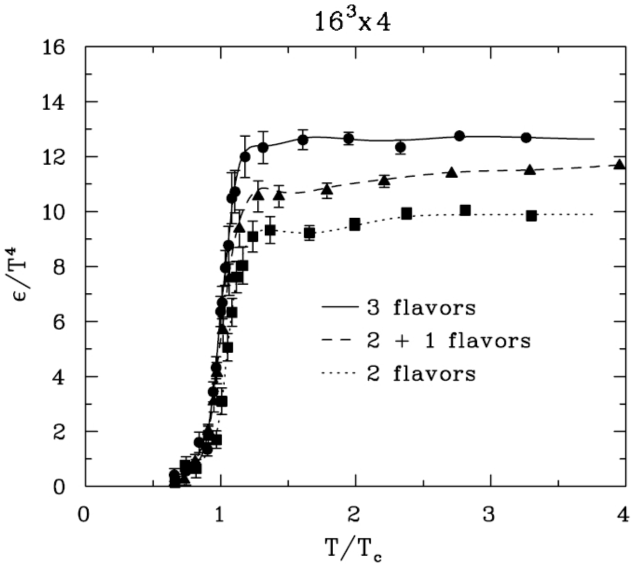

Figure 11: The energy density in units of T^4 versus T/T_c for two, three flavour of light quarks and for two flavour of light and one of heavy quarks. This graph is taken from [1] and the data from [7]

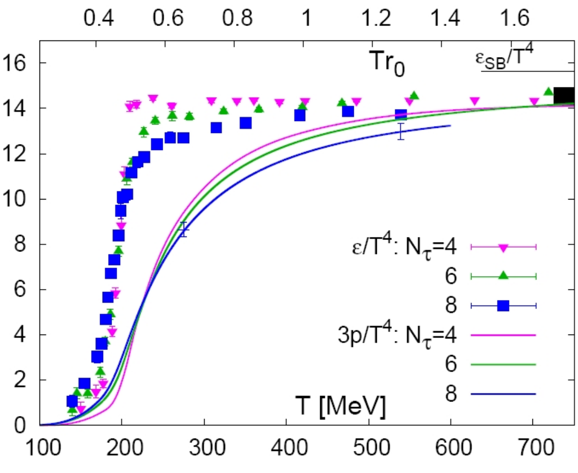

Figure 12: The energy density and three times the pressure calculated in the physical point. Graph taken from [9]

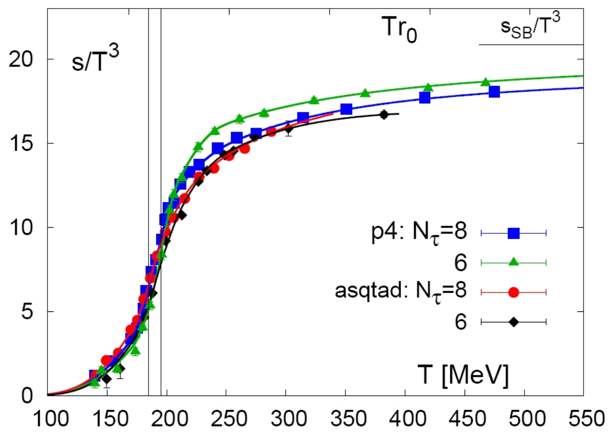

Figure 13: The entropy density in function of the temperature in the physical point, taken from [9]

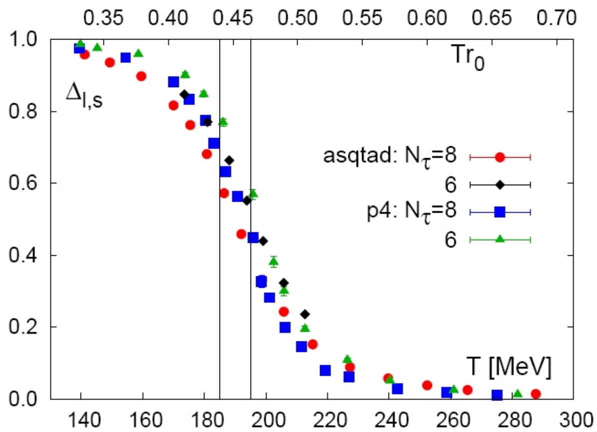

Figure 14: The subtracted chiral condensate vs. the temperature in the physical point. The graph is taken from [9]

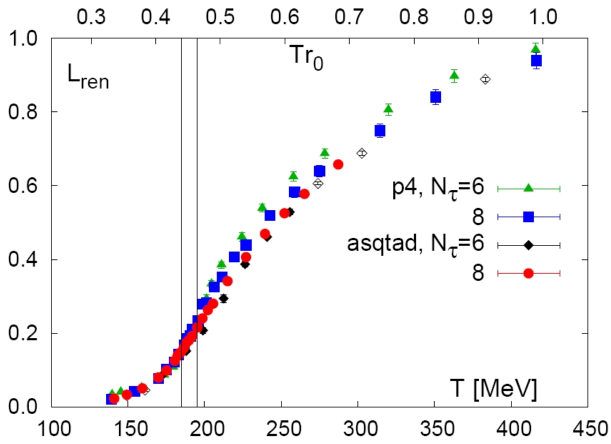

Figure 15: The Polyakov loop in function of the temperature in the physical point, taken from [9]

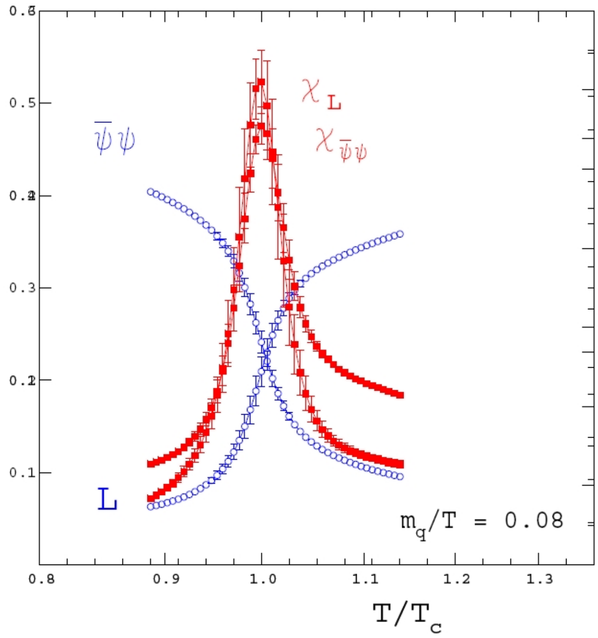

Figure 16: The chiral condensate $\langle\bar{\psi}\psi\rangle$ and the Polyakov loop vs. temperature in the physical point. The fact that the critical temperature is about the same for both transitions is not well understood yet. The graph is taken from [3]

A Appendix

A.1 Calculation of the pressure of an ideal gas of bosons or fermions

We begin using the partition function found in [13]

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left(-\frac{\beta p}{2} - \ln\left(1 \pm e^{-\beta p}\right) \right). \tag{32}$$

Using the thermodynamical relations we can calculate the pressure as

$$P = \frac{\partial (T \ln Z)}{\partial V}. \tag{33}$$

Then the first term in the integral can be neglected because in the pressure it will be only a constant. We now define

$$I = \int \frac{d^3p}{(2\pi)^3} \ln\left(1 \pm e^{-\beta p}\right). \tag{34}$$

Let us now calculate

$$\begin{aligned}
\frac{\partial I}{\partial \beta} &= \int \frac{d^3p}{(2\pi)^3} \frac{\mp p e^{-\beta p}}{1 \pm e^{-\beta p}} && (35)\\
&= \mp \int \frac{d^3p}{(2\pi)^3} \left(\pm p \mp \frac{p}{1 \pm e^{-\beta p}} \right) && (36)\\
&= \frac{4\pi}{(2\pi)^3} \int dp \frac{p^3}{1 \pm e^{-\beta p}} && (37)\\
&= \begin{cases} \frac{\pi^2}{30\beta^4}, & \text{for bosons} \\ \frac{7}{8}\frac{\pi^2}{30\beta^4}, & \text{for fermions} \end{cases} . && (38)
\end{aligned}$$

where we dropped out, one more, the constant term which will compare in the pressure. Then for bosons

$$I = -\frac{\pi^2}{90} T^3. \tag{39}$$

It follow that the pressure of the ideal gas of bosons is

$$P = \frac{\pi^2}{90} T^4. \tag{40}$$

References

[1] J. I. Kapusta and C. Gale, “Finite-temperature field theory: Principles and applications,” *Cambridge, UK: Univ. Pr.* (2006) 428 p

[2] J. B. Kogut and M. A. Stephanov, “The phases of quantum chromodynamics: From confinement to extreme environments,” *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **21** 1 (2004)

[3] Hands S., The Phase Diagram of QCD, arXiv: physics/0105022v1 (2001)

[4] Boyd G., Engels J., Karsch F., Laermann E., Legeland C., Luetgemeier M., and Petersson B., *Nucl. Phys. B* **469**, 419 (1996)

[5] Kaczmarek O., Karsch, F., Petreczky P., and Zantow F., *Phys. Lett. B* **543**, 41 (2002)

[6] Kaczmarek O., Karsch, F., Zantow F., and Petreczky P., *Phys. Rev. D* **70**, 074505 (2004)

[7] Karsch F., Laermann E., and Peikert A., *Phys. Lett. B* **478**, 447 (2000)

[8] Bernard, C., et al. (MILC collaboration), *Nucl. Phys. Proc. Suppl.* **119**, 523 (2003)

[9] Bazavov A., et al. Equation of state and QCD transition at finite temperature, arXiv: 0903.4379v1 (2009)

[10] Pisarski D., Wilczek F. Remarks on the chiral transition in chromodynamics **29**, 338 (1984)

[11] Traub F., Goldstone’s Theorem and Chiral Symmetry Breaking (2009)

[12] Schneider B., Lattice formulation of Yang-Mills theory and confinement at strong coupling (2009)

[13] Mani R., Finite Temperature QCD: formulation and symmetries (2009)