Finite temperature QCD: formulation and symmetries

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1 Finite temperature in the Euclidean path integral

1.1 Partition function of the grand canonical ensemble

The grand canonical ensemble is used to describe systems in contact with a heat reservoir at temperature T and a particle reservoir. The system can exchange energy as well as particles with these reservoirs. In this ensemble the temperature, the chemical potential and the volume are kept fixed. In a relativistic quantum system, where particles can be destroyed and created, it is most straightforward to compute observables in the grand canonical ensemble. Consider a grand canonical ensemble described by a Hamiltonian H and n conserved charges $\hat{N}_i$, $i = 1, \cdots, n$, which commute with H. From statistical quantum mechanics we know that the density matrix of the grand canonical ensemble is given by

$$\hat{\rho} = exp\left[-\beta(H - \mu_i \hat{N}_i)\right] \tag{1}$$

where we sum over i. The inverse temperature is given by $\beta = T^{-1}$ and μ_i are chemical potentials corresponding to the conserved charges. The density matrix is the operator, which is diagonal with respect to the basis $\{|n\rangle\}$ where $|n\rangle$ are energy eigenstates. The eigenvalues are the relative probabilities for the occurrence of the n'th eigenstate.

The partition function of the system is then given by

$$Z = \mathrm{Tr}\hat{\rho} \tag{2}$$

and the ensemble average of an operator $\hat{A}$ is

$$\langle A \rangle = \frac{\mathrm{Tr}\hat{\rho}\hat{A}}{\mathrm{Tr}\hat{\rho}} \tag{3}$$

The trace operation means that we sum over all diagonal matrix elements of an operator in a given basis of the Hilbert space the operator is acting on. Let $\{|n\rangle\}$ be a basis of the Hilbert space and $\hat{A}$ an operator acting on that space, then

$$\mathrm{Tr}\hat{A} = \sum_n \langle n|\hat{A}|n\rangle \tag{4}$$

If we consider a non-discrete set of basis elements, we would replace the sum by an integral. The partition function is the most important function in thermodynamics because it contains all thermodynamic information like pressure, entropy, particle number and energy given by

$$\begin{aligned} P &= \frac{\partial(T \ln Z)}{\partial V} \\ N_i &= \frac{\partial(T \ln Z)}{\partial \mu_i} \end{aligned}$$

$$\begin{aligned} S &= \frac{\partial(T\ln Z)}{\partial T} \\ E &= -PV + TS + \mu_i N_i \\ F &= -T\ln Z \end{aligned} \tag{5}$$

1.2 Path integral representation of the partition function for bosons

We want to derive a path integral representation of the partition function for bosonic systems in quantum mechanics. Let $q = \{q_\alpha\}$ denote the coordinate degrees of freedom of the system and $|q\rangle$ the simultaneous eigenstates of the corresponding operators $\{\hat{q}_\alpha\}$, i.e.

$$\hat{q}_\alpha |q\rangle = q_\alpha |q\rangle, \quad \alpha = 1, \cdots, k \tag{6}$$

The partition function is given by

$$Z = \mathrm{Tr} e^{-\beta H} = \int dq \langle q| e^{-\beta H} |q\rangle \tag{7}$$

where the integration measure is

$$dq = \prod_{\alpha=1}^{k} dq_\alpha \tag{8}$$

and the Hamiltonian H is a function of the position and momentum operators. We will specify the structure of the Hamiltonian later on. The trace here means that we integrate over q, which we will denote by $q^{(0)}$ in the following. We now set $\beta = \epsilon N$ to write

$$Z = \int dq \langle q| \underbrace{e^{-\epsilon H} e^{-\epsilon H} \cdots e^{-\epsilon H}}_{N \text{ times}} |q\rangle \tag{9}$$

and insert a complete set of states in between of each exponential:

$$\int \prod_{i=0}^{N-1} dq^{(i)} \langle q^{(N)}| e^{-\epsilon H} |q^{(N-1)}\rangle \langle q^{(N-1)}| e^{-\epsilon H} |q^{(N-2)}\rangle \cdots \langle q^{(1)}| e^{-\epsilon H} |q^{(0)}\rangle \big|_{q^{(0)}=q^{(N)}} \tag{10}$$

For Hamiltonians of the form[1]

$$H = \frac{1}{2} \sum_{\alpha=1}^{k} \hat{p}_\alpha^2 + V(\hat{q}) \tag{11}$$

with the momenta $\hat{p}_\alpha$ canonically conjugated to $\hat{q}_\alpha$, we can evaluate the matrix elements appearing in (10). Using the Baker-Hausdorff formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]\cdots}$ we can approximate the following matrix elements (ϵ small)

$$\langle q^{(i+1)}| e^{-\epsilon H} |q^{(i)}\rangle \approx \langle q^{(i+1)}| e^{-\epsilon/2 \sum_\alpha \hat{p}_\alpha^2} |q^{(i)}\rangle e^{-\epsilon V(q^{(i)})} \tag{12}$$

[1] Actually, the following calculation holds for any Hamiltonian of the form $H(\hat{p},\hat{q}) = P_1(\hat{p}) + P_2(\hat{q})$ where P_i are polynomials.

To evaluate the remaining matrix element we insert, similarly as before, a complete set of momentum eigenstates to the right of the exponential. With

$$\langle q|p\rangle = \prod_{\alpha=1}^{k} \frac{1}{\sqrt{2\pi}} e^{ip_\alpha q_\alpha} \tag{13}$$

we have

$$\langle q^{(i+1)}|e^{-\epsilon H}|q^{(i)}\rangle \approx e^{-\epsilon V(q^{(i)})} \int dp^{(i)} \prod_{\alpha=1}^{k} \exp\left\{-\epsilon\left[\frac{1}{2}p_\alpha^{(i)2} - ip_\alpha^{(i)}\left(\frac{q_\alpha^{(i+i)} - q_\alpha^{(i)}}{\epsilon}\right)\right]\right\} \tag{14}$$

with the integration measure

$$dp^{(i)} = \prod_{\delta=1}^{k} \frac{dp_\delta^{(i)}}{2\pi} \tag{15}$$

Putting all together we arrive at the path integral form of the partition function

$$Z = \lim_{N\to\infty \epsilon\to 0 N\epsilon=\beta} \int Dq \int Dp\, e^{\sum_{i=0}^{N-1}\sum_\alpha ip_\alpha^{(i)}(q_\alpha^{(i+1)} - q_\alpha^{(i)}) - \epsilon H(q^{(i)},p^{(i)})}\Big|_{q^{(N)}=q^{(0)}} \tag{16}$$

with the measure

$$DqDp = \prod_{i=0}^{N-1}\prod_\alpha \frac{dq_\alpha^{(i)} dp_\alpha^{(i)}}{2\pi} \tag{17}$$

and the formal continuum limit

$$\int_{periodic} Dq \int Dp\, e^{\int_0^\beta d\tau [\sum_\alpha ip_\alpha(\tau)\dot{q}_\alpha(\tau) - H(q(\tau),p(\tau))]} \tag{18}$$

Note that the Hamiltonian in (16) is now a function of the eigenvalues of the momentum and position operators. Before, it was a function of the operators. The subscript "periodic" means that the coordinates at $\tau = 0$ and $\tau = \beta$ have to be identified (periodic boundary conditions). If we also consider conserved charges, we simply make the replacement $H \to H - \mu_i N_i$.

The generalization to field theory is immediate. We replace $q_\alpha(\tau)$ by a scalar field $\phi(\vec{x}, \tau)$ and $p_\alpha(\tau)$ by the conjugate momentum $\pi(\vec{x}, \tau)$ of the field. That means, that we go over to a system with infinitely many degrees of freedom. $\vec{x}$ labels these degrees of freedom and replaces the label α. The Hamilonian H is now given by

$$H = \int d^3x \mathcal{H}(\phi, \pi) \tag{19}$$

where $\mathcal{H}$ is the Hamiltonian density. The equivalent path integral form for fields is

$$Z = \int [\mathrm{d}\pi] \int_{periodic} [\mathrm{d}\phi] \times \exp\Big[\textstyle\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \Big(\mathrm{i}\pi \frac{\partial\phi(\mathbf{x},\tau)}{\partial\tau} - \mathcal{H}(\pi, \phi) + \mu\mathcal{N}(\pi, \phi)\Big)\Big] \tag{20}$$

where

$$[\mathrm{d}\pi][\mathrm{d}\phi] = \lim_{N\to\infty}\left(\prod_{i=1}^{N}\mathrm{d}\pi_i\mathrm{d}\phi_i/2\pi\right) \tag{21}$$

The conserved charge density $\mathcal{N}$ is a consequence of Noether‘s theorem, which states that whenever a Lagrangian has a global continuous symmetry there is an associated conserved current.
The most important point here to notice is that the integration $\int_{periodic}[\mathrm{d}\phi]$ over the field is constrained such that the field is periodic in imaginary time: $\phi(\mathbf{x},0) = \phi(\mathbf{x},\beta)$.
This means that we have compactified euclidean space-time because euclidean space-time now has a finite extension in the time direction. We can imagine cutting space-time (in two dimensions) at $\tau = 0$ and $\tau = \beta$ and stick both ends of this strip together such that we are left with a cylinder.

1.3 Partition function for systems with $\mathcal{H}$ quadratic in π

The phase-space path integral (20) contains integrations over momenta. If the Hamiltonian density is quadratic in these momenta, we can explicitly integrate over these momenta. This leads to a configuration space path integral. Consider a Hamiltonian density of the form

$$\mathcal{H} = \pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi) \tag{22}$$

with potential $U(\phi)$ describing interactions. Discretizing the partition function and plugging in (22) yields:

$$Z = \lim_{N\to\infty}\left(\prod_{i=1}^{N}\int_{-\infty}^{\infty}\frac{\mathrm{d}\pi_i}{2\pi}\int_{periodic}\mathrm{d}\phi_i\right)\exp\Big\{\sum_{j=1}^{N}\int\mathrm{d}^3x\Big[\mathrm{i}\pi_j(\phi_{j+1}-\phi_j)$$
$$-\Delta\tau\left(1/2\pi_j^2 + 1/2(\nabla\phi_j)^2 + 1/2m^2\phi_j^2 + U(\phi_j)\right)\Big]\Big\} \tag{23}$$

Because of the position space integration in the exponent we must replace $\int \mathrm{d}^3x$ by a sum over little cubes. This means that we discretize position space as well. We divide it into M^3 little cubes of length a. Then $V = L^3$ and $L = aM$. ($a \to 0$, $M \to \infty$)
We then have

$$Z = \lim_{N\to\infty}\left(\prod_{i=1}^{N}\int_{-\infty}^{\infty}\frac{\mathrm{d}\pi_i}{2\pi}\int_{periodic}\mathrm{d}\phi_i\right)\exp\Big\{\sum_{j=1}^{N}\sum_{i=1}^{M^3}a^3\Big[\mathrm{i}\pi_j(\phi_{j+1}-\phi_j)$$
$$-\Delta\tau\left(1/2\pi_j^2 + 1/2(\nabla\phi_j)^2 + 1/2m^2\phi_j^2 + U(\phi_j)\right)\Big]\Big\} \tag{24}$$

Since the momentum integrations are products of gaussian integrals we can use the following formula to proceed

$$\int_{\mathbb{R}^N} \exp(-\frac{1}{2}\mathbf{x}A\mathbf{x} + \mathrm{i}\xi\mathbf{x})\mathrm{d}^N x = (2\pi)^{N/2}(\det A)^{-1/2} \exp\left[-\frac{1}{2}\xi A^{-1}\xi\right] \tag{25}$$

In our case A is the $N \times N$ identity matrix times $a^3\Delta\tau$ and $\xi_j = a^3(\phi_{j+1} - \phi_j)$. Applying this formula to (24) yields

$$Z \propto \lim_{N,M\to\infty} (2\pi)^{-M^3N/2} \left(\prod_{i=1}^{N} \int_{periodic} \mathrm{d}\phi_i\right) \exp\Big\{\Delta\tau \sum_{j=1}^{N} \int \mathrm{d}^3x \Big[-\frac{1}{2}\left(\frac{\phi_{j+1} - \phi_j}{\Delta\tau}\right)^2 - 1/2(\nabla\phi_j)^2 - 1/2m^2\phi_j^2 - U(\phi_j)\Big]\Big\} \tag{26}$$

Going to the continuum limit we finally arrive at the expression

$$Z = N' \int_{periodic} [\mathrm{d}\phi] \exp\left(-\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2 + 2U(\phi_j)\right)\right] \tag{27}$$

or with $\mathcal{L}_E = \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + U(\phi_j)$ the euclidean Lagrangian density

$$Z = N' \int_{periodic} [\mathrm{d}\phi] \exp\left(-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\mathcal{L}_E\right) := N' \int_{periodic} [\mathrm{d}\phi] e^{-S_E} \tag{28}$$

In this expression, the constant N' is irrelevant because the multiplication of the partition function by a constant does not change the thermodynamics. The expectation values are only shifted by a constant. Furthermore, we have defined the exponent in (28) to be S_E, the euclidean action.

This is a remarkable result. The partition function is a weighted sum over all field configurations that live on a euclidean space time surface compactified along the time direction, the cylinder mentioned earlier. The radius of this cylinder gets bigger and bigger as the temperature is lowered.

To evaluate the remaining path integral we can expand the field and integrate S_E by parts[2]. We now consider a non interacting field ($U(\phi) = 0$)

$$S_E = \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \phi\left(-\frac{\partial^2}{\partial\tau^2} - \nabla^2 + m^2\right)\phi \tag{29}$$

$$\phi(\mathbf{x},\tau) = \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} e^{\mathrm{i}(\mathbf{p}\mathbf{x}+\omega_n\tau)}\phi_n(\mathbf{p}) \tag{30}$$

[2]Because of the periodicity of ϕ, the boundary terms vanish

Because the field is periodic in time, the temporal Fourier transform turns into a Fourier series. The sum over the momenta $\mathbf{p}$ can be interpreted as a discretized Fourier transform.[3] We will later go back to the continuum.
Because of the periodicity $\phi(\mathbf{x}, \beta) = \phi(\mathbf{x}, 0)$ we have to set $\omega_n = 2\pi n/\beta = 2\pi nT$. These frequencies are called Matsubara frequencies. Plugging (30) into (29) gives

$$S_E = \frac{\beta}{V}\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \sum_{n,n'=-\infty}^{\infty}\sum_{\mathbf{p},\mathbf{p}'} e^{\mathrm{i}(\mathbf{px}+\omega_n\tau)}(\omega_n^2+\omega^2)\phi_n(\mathbf{p})\phi_{n'}(\mathbf{p}')e^{\mathrm{i}(\mathbf{p}'\mathbf{x}+\omega_{n'}\tau)} \tag{31}$$

with $\omega = (\mathbf{p}^2 + m^2)^{1/2}$. The field is real, such that $\phi_{-n}(-\mathbf{p}) = \phi_n^*(\mathbf{p})$. Equation (31) can thus be evaluated further[4]

$$S_E = \frac{1}{2}\beta^2 \sum_{n=-\infty}^{\infty}\sum_{\mathbf{p}}(\omega_n^2+\omega^2)\phi_n(\mathbf{p})\phi_n^*(\mathbf{p}) \tag{32}$$

The term in brackets in the sum will turn out to be the inverse propagator in frequency momentum space (see section 1.5). We again use (25) with $A = \beta^2(\omega_n^2 + \omega^2)$ to arrive at

$$Z = \prod_n \prod_{\mathbf{p}} \left[\beta^2(\omega_n^2+\omega^2)\right]^{-1/2} \tag{33}$$

Using some mathematical tricks and neglecting a β independent piece we can express the logarithm of the partition function as

$$\ln Z = \sum_{\mathbf{p}}\left(-\frac{\beta\omega}{2} - \ln(1-e^{-\beta\omega})\right) \tag{34}$$

or going to the continuum

$$\ln Z = V\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(-\frac{\beta\omega}{2} - \ln(1-e^{-\beta\omega})\right) \tag{35}$$

This is the known expression for $\ln Z$ for bosons but including the zero-point energy.

1.4 Fermions

In this section we first look at a simple example of a fermionic system with one degree of freedom. The main difficulty is that creation and annihilation operators for fermions satisfy anticommutation relations in contrast to commutation relations for bosons. Consider a system whose Hilbert space only consists of the vacuum state $|0\rangle$ and the one particle state $|1\rangle = \hat{a}^\dagger|0\rangle$. $|0\rangle$ is annihilated by the operator $\hat{a}$. The creation and annihilation operators,

[3] We could also have put an integral, it is just more convenient to evaluate the partition function.
[4] The integrals vanish unless $\mathbf{p} = -\mathbf{p}'$ and $n = -n'$. In the latter case the value of the integral is β/V

$\hat{a}, \hat{a}^\dagger$ satisfy the anticommutation relation $\{\hat{a}, \hat{a}^\dagger\} = 1$, whereas all other anticommutators vanish. A general operator acting on this space has the form

$$\hat{A} = K_{00} + K_{10}\hat{a}^\dagger + K_{01}\hat{a} + K_{11}\hat{a}^\dagger\hat{a} \tag{36}$$

The K_{ij} are not the matrix elements of $\hat{A}$. The matrix is given by

$$\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{00} + K_{11} \end{pmatrix} \tag{37}$$

We need an expression for the trace of an operator and the product of operators, in terms of an integral over Grassmann variables. Define the generators a and a^* of a Grassmann algebra satisfying $\{a, a^*\} = \{a, a\} = \{a^*, a^*\} = 0$.
We replace the operators $\hat{a}$ and $\hat{a}^\dagger$ in (36) by the generators a and a^* and multiply with $e^{a^*a} = 1 + a^*a$ to get the matrix form of $\hat{A}$

$$\begin{aligned} A(a^*, a) &= e^{a^*a}(K_{00} + K_{10}a^* + K_{01}a + K_{11}a^*a) \\ &= (K_{00} + K_{10}a^* + K_{01}a + K_{11}a^*a) + K_{00}a^*a \\ &= A_{00} + A_{10}a^* + A_{01}a + A_{11}a^*a \end{aligned} \tag{38}$$

where the A_{ij} now really are the matrix elements of $\hat{A}$. We also define the normal form

$$\tilde{A}(a^*, a) = K_{00} + K_{10}a^* + K_{01}a + K_{11}a^*a \tag{39}$$

Next, we find using the Grassmann integration rules that

$$\mathrm{Tr}\hat{A} = \int da da^* e^{a^*a} A(a^*, a) \tag{40}$$

or equivalently

$$\mathrm{Tr}\hat{A} = \int da^* da e^{-a^*a} A(a^*, -a) \tag{41}$$

The matrix elements of the product $\hat{C} = \hat{A}\hat{B}$ are given by the coefficients of

$$C(a^*, a) = \int \mathrm{d}b^* \mathrm{d}b\, e^{-b^*b} A(a^*, b) B(b^*, a) \tag{42}$$

where $a, a^*, b, b^*, \mathrm{d}b, \mathrm{d}b^*$ are all anticommuting variables. Eq. (41) gives rise to antiperiodic boundary conditions in the path integral form of the partition function. We will see that later.
We can also have two sets of operators $\{\hat{a}_i\}$ and $\{\hat{a}_i{}^\dagger\}$ $(i = 1, 2, \cdots, n)$ with $\{\hat{a}_i, \hat{a}_i^\dagger\} = \delta_{ij}$ (all other anticommutators vanish) and the associated Grassmann generators $\{a_i\}$ and $\{a_i^*\}$. with $\{a_i, a_j^*\} = \{a_i, a_j\} = \{a_i^*, a_j^*\} = 0$, $(i, j = 1, \cdots, n)$.

Now we are ready to find the path integral representation of the grand canonical partition function for fermions in a simple model. We consider a Hamiltonian of the form

$$\hat{H} = E\,\hat{a}^\dagger \hat{a} \tag{43}$$

and a number operator

$$\hat{N} = \hat{a}^\dagger \hat{a} \tag{44}$$

and divide the time interval $[0, \beta]$ into N intervals of length $\epsilon = \beta/N$ to write

$$Z = \mathrm{Tr}\hat{\rho} = \mathrm{Tr}e^{-\beta(\hat{H}-\mu\hat{N})} = \mathrm{Tr}\,(e^{-\epsilon(\hat{H}-\mu\hat{N})})^N = \mathrm{Tr}\,(e^{-\epsilon(E-\mu)\hat{a}^\dagger\hat{a}})^N := \mathrm{Tr}\,\hat{\rho}_\epsilon^N \tag{45}$$

The matrix form of $\hat{\rho}_\epsilon$ is given by

$$\rho_\epsilon(a^*, a) = e^{a^*a}\tilde{\rho}_\epsilon(a^*, a) = e^{a^*a}e^{(e^{-\epsilon(E-\mu)}-1)a^*a} \approx e^{a^*a}e^{-\epsilon(E-\mu)a^*a} \tag{46}$$

where the approximation is valid for small ϵ. Using the fact that $e^{a_i^*a_j}$ commutes with $e^{a_m^*a_n}$ for arbitrary i, j, m, n (and also $e^{a_i^*a_j}e^{a_m^*a_n} = e^{a_i^*a_j+a_m^*a_n}$), we can write

$$\rho_\epsilon(a_i^*, a_{i-1}) = e^{a_i^*a_{i-1}(1-\epsilon(E-\mu))} \tag{47}$$

We can calculate the matrix form $\rho(a_N^*, a_N)$ of $\hat{\rho}_\epsilon^N$ using the formula for the products:

$$\begin{aligned}
\rho(a_N^*, a_N) &= \int \prod_{i=1}^{N-1} da_i^* da_i e^{-a_i^*a_i}\rho_\epsilon(a_N^*, a_{N-1})\rho_\epsilon(a_{N-1}^*, a_{N-2})\cdots\rho_\epsilon(a_1^*, a_0)|_{a_0=a_N} \\
&= \int \prod_{i=1}^{N-1} da_i^* da_i \left(e^{a_N^*a_{N-1}(1-\epsilon(E-\mu))}\right) e^{a_i^*a_{i-1}(1-\epsilon(E-\mu))-a_i^*a_i}|_{a_0=a_N}
\end{aligned} \tag{48}$$

Finally we compute the trace to get

$$Z = \mathrm{Tr}\hat{\rho} = \int da_N^* da_N e^{-a_N^*a_N}\rho(a_N^*, -a_N) = \int \prod_{i=1}^{N} da_i^* da_i e^{a_i^*a_{i-1}(1-\epsilon(E-\mu))-a_i^*a_i}|_{a_0=-a_N} \tag{49}$$

Here, we see that the integration is such that $a_0 = -a_N$, which means that we have antiperiodic boundary conditions. After relabeling $a_i \to a_{i+1}$ and rewriting, we get

$$Z = \int \prod_{i=1}^{N} da_i^* da_i e^{-a_i^*(a_{i+1}-a_i)-\epsilon(E-\mu)a_i^*a_i}|_{a_1=-a_{N+1}} \tag{50}$$

which reads after going to the continuum ($N \to \infty, \epsilon \to 0, N\epsilon = \beta$)

$$\int_{antip} [da^*][da]e^{-\int_0^\beta d\tau(a^*\frac{\partial a}{\partial \tau}+H-\mu N)} \tag{51}$$

where $H(a^*, a) = E\, a^* a$ is the Hamiltonian and $N(a^*, a) = a^* a$ is the conserved charge. This expression can be generalized to arbitrarily many degrees of freedom and to field theory. We replace the Grassmann variables by Grassmann valued fields $\psi_\alpha(\vec{x}, \tau)$ and $\psi^*_\alpha(\vec{x}, \tau)$ where $\vec{x}$ labels the infinitely many degrees of freedom. The corresponding path integral for fields reads

$$\int_{antiper} [d\psi^*][d\psi] \exp\left[-\int_0^\beta d\tau \int d^3x \left(\psi^*(x)\frac{\partial}{\partial\tau}\psi(x) + \mathcal{H}(\psi, \psi^*) - \mu\mathcal{N}(\psi, \psi^*)\right)\right] \quad (52)$$

As an example, we consider the Hamiltonian density for the free Dirac field

$$\mathcal{H} = \bar{\psi}(\vec{\gamma}^E \cdot \nabla + m)\psi \quad (53)$$

and the corresponding euclidean Lagrangian density

$$\mathcal{L}_E = \bar{\psi}(\partial\!\!\!/ + m)\psi \quad (54)$$

where $\partial\!\!\!/ = \gamma^E_\mu \partial_\mu$ and $\bar{\psi} = \psi^* \gamma^E_4$. The euclidean gamma matrices are related to the Dirac gamma matrices through $\gamma^E_4 = \gamma^0, \gamma^E_i = -\mathrm{i}\gamma^i$ satisfying $\{\gamma^E_\mu, \gamma^E_\nu\} = 2\delta_{\mu\nu}$. This Lagrangian has a global U(1) symmetry. It is invariant under $\psi \to \psi e^{-\mathrm{i}\alpha}$ and the associated Noether current is $j_\mu = \bar{\psi}\gamma^E_\mu \psi$. Since $(\gamma^E_4)^2 = 1$, we find the conserved charge density

$$\mathcal{N} = j_4 = \psi^*\psi \quad (55)$$

The partition function can again be expressed as a path integral

$$Z = \int_{antiper} [d\psi^*][d\psi] \exp\left[-\int_0^\beta d\tau \int d^3x\, \psi^* \gamma^E_4 \left(\gamma^E_4 \frac{\partial}{\partial\tau} + \vec{\gamma}^E \cdot \nabla + m - \mu\gamma^E_4\right)\psi\right] \quad (56)$$

In order to calculate the path integral (56) we expand $\Psi(\mathbf{x}, \tau)$ like we did for bosons.

$$\psi_\alpha(\mathbf{x}, \tau) = (1/V)^{1/2} \sum_n \sum_{\mathbf{p}} e^{\mathrm{i}(\mathbf{p}\cdot\mathbf{x} + \omega_n\tau)} \psi_{\alpha;n}(\mathbf{p}) \quad (57)$$

where α is a Lorentz index. Here we set the Matsubara frequencies $\omega_n = (2n+1)\pi/\beta$ because of the antiperiodicity of the fields. We can now insert (57) into (56) and we get

$$\begin{aligned} Z &= \left[\prod_n \prod_{\mathbf{p}} \prod_\alpha \int d\psi^*_{\alpha;n}(\mathbf{p})\, d\psi_{\alpha;n}(\mathbf{p})\right] e^{-S_E} \\ S_E &= \sum_n \sum_{\mathbf{p}} \psi^*_{\alpha;n}(\mathbf{p}) D_{\alpha\beta} \psi_{\beta;n}(\mathbf{p}) \\ D &= \beta[(\mathrm{i}\omega_n - \mu) + \mathrm{i}\gamma^E_4 \vec{\gamma}^E \cdot \mathbf{p} + m\gamma^E_4] \end{aligned} \quad (58)$$

where S_E is the euclidean action in frequency momentum space.
The Grassmann valued fields anti-commute among each other. To handle this situation we use an important formula

$$\int \mathrm{d}\eta_1^\dagger \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_N^\dagger \mathrm{d}\eta_N e^{\eta^\dagger D \eta} = \det D \tag{59}$$

Here η_i and $\eta_i^\dagger$ are all Grassmann variables and D is a $N \times N$ matrix.
We can now evaluate (58) which simply gives:

$$Z = \det(-D) \tag{60}$$

In the end we get the result

$$\ln Z = 2V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega + \ln\left(1 + e^{-\beta(\omega-\mu)}\right) + \ln\left(1 + e^{-\beta(\omega+\mu)}\right)\right] \tag{61}$$

where the $-\mu$ term corresponds to contributions from antiparticles and the μ term to those from particles. There are two main differences between bosons and fermions. First, for fermions, we must integrate over Grassmann variables. Therefore $Z = \det D$ in contrast to $Z = (\det D)^{-1/2}$. This difference is responsible for the Bose-Einstein factor $1 - e^{-\beta\omega}$ in contrast to the Fermi factor $1 + e^{-\beta\omega}$.
The second main difference, is that fermion fields are antiperiodic in euclidean time, whereas bosonic fields are periodic.

1.5 Thermal Green‘s functions and propagators in finite temperature field theory

Let $\phi(x)$ be a real scalar field whose dynamics is governed by a Hamiltonian H. We know that at zero temperature, the Green's functions are given by the ground state expectation value of time ordered products of the field operators $\hat{\phi}(x)$. The analogue at finite temperature are the thermal Green‘s functions defined by

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = Z^{-1} \mathrm{Tr}[e^{-\beta H} T(\hat{\phi}(x_1) \cdots \hat{\phi}(x_n))] \tag{62}$$

which reduces to the zero temperature Green‘s functions in the limit $\beta \to \infty$. The interpretation is that all eigenstates of the Hamiltonian are excited with a probability given by the Boltzmann factor when the system is placed in contact with a heat bath. The expression for the Green‘s function for fermions is almost the same. We only replace the field operators $\hat{\phi}$ by fermionic field operators $\hat{\psi}$ and define the time ordering operator T as follows:

$$T(\hat{\psi}(\tau_1)\hat{\psi}(\tau_2)) = \hat{\psi}(\tau_1)\hat{\psi}(\tau_2)\theta(\tau_1 - \tau_2) - \hat{\psi}(\tau_2)\hat{\psi}(\tau_1)\theta(\tau_2 - \tau_1) \tag{63}$$

Examining the Green's functions for $\hat{\phi}$ and $\hat{\psi}$ directly verifies that the fields are periodic or antiperiodic respectively. We can use the cyclic property of the trace and the imaginary Heisenberg time evolution $\hat{\phi}(\mathbf{y},\beta) = e^{\beta H}\hat{\phi}(\mathbf{y},0)e^{-\beta H}$ to find

$$\begin{aligned}
\langle \phi(\mathbf{x},\tau)\phi(\mathbf{y},0)\rangle &= Z^{-1}\mathrm{Tr}[e^{-\beta H}\hat{\phi}(\mathbf{x},\tau)\hat{\phi}(\mathbf{y},0)] \\
&= Z^{-1}\mathrm{Tr}[\hat{\phi}(\mathbf{y},0)e^{-\beta H}\hat{\phi}(\mathbf{x},\tau)] \\
&= Z^{-1}\mathrm{Tr}[e^{-\beta H}e^{\beta H}\hat{\phi}(\mathbf{y},0)e^{-\beta H}\hat{\phi}(\mathbf{x},\tau)] \\
&= Z^{-1}\mathrm{Tr}[e^{-\beta H}\hat{\phi}(\mathbf{y},\beta)\hat{\phi}(\mathbf{x},\tau)] \\
&= Z^{-1}\mathrm{Tr}[e^{-\beta H}\mathrm{T}(\hat{\phi}(\mathbf{y},\beta)\hat{\phi}(\mathbf{x},\tau)]) \\
&= \langle \phi(\mathbf{x},\tau)\phi(\mathbf{y},\beta)\rangle
\end{aligned} \tag{64}$$

This implies that

$$\phi(\mathbf{y},0) = \phi(\mathbf{y},\beta) \tag{65}$$

The same calculations give $\psi(\mathbf{y},0) = -\psi(\mathbf{y},\beta)$ for fermionic fields.
The thermal Green's functions can be obtained from a generating functional

$$Z[J] = \int_{periodic} [\mathrm{d}\phi]\, e^{-S_E + \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, J\phi} \tag{66}$$

by taking functional derivatives with respect to the source J at $J = 0$. An expectation value of an operator can be expressed as

$$\langle \mathcal{O}\rangle = \frac{\int_{periodic}[\mathrm{d}\phi]\, \mathcal{O}(\phi)e^{-S_E}}{Z[J=0]} \tag{67}$$

For the free neutral scalar field, (66) can be integrated (gaussian integral):

$$Z[J] = Z[0]e^{\frac{1}{2}\int_\beta \mathrm{d}^4x \int_\beta \mathrm{d}^4y\, J(x)\Delta(x-y)J(y)} \tag{68}$$

where

$$\int_\beta \mathrm{d}^4x = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \tag{69}$$

and $\Delta(x-y)$ is the inverse operator (the propagator) of $-\left(\frac{\partial}{\partial\tau}\right)^2 - \nabla^2 + m^2$ or simply the thermal Green's function $\langle\phi(x),\phi(y)\rangle$. The propagator is periodic in the euclidean time direction

$$\Delta(\vec{x},0) = \Delta(\vec{x},\beta) \tag{70}$$

and therefore also has a Fourier expansion

$$\Delta(x) = \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3}\tilde{\Delta}(\omega_n,\mathbf{p})e^{\mathrm{i}(\omega_n\tau + \mathbf{p}\cdot\vec{x})} \tag{71}$$

The propagator in frequency momentum space is found to be

$$\tilde{\Delta}(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2} \tag{72}$$

and

$$\Delta(x) = \frac{1}{\beta} \sum_l \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{\mathrm{i}(\omega_n \tau + \mathbf{p}\cdot\vec{x})}}{\omega_n^2 + \mathbf{p}^2 + m^2} \tag{73}$$

When we compare this to the $T = 0$ propagator in euclidean space time[5]

$$\Delta(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{\mathrm{i}p\cdot z}}{p^2 + m^2} \tag{74}$$

this suggests that we can get, in general, finite temperature expressions from those at $T = 0$ by making the substitutions

$$\begin{aligned} p_4 &\rightarrow \omega_n \quad \omega_n = 2\pi n/\beta \\ \int \frac{\mathrm{d}p_4}{2\pi} f(p_4) &\rightarrow \frac{1}{\beta} \sum_n f(\omega_n) \\ \int \mathrm{d}^4 x &\rightarrow \int_\beta \mathrm{d}^4 x \end{aligned} \tag{75}$$

We have seen that in the action for fermions, the chemical potential appears in the kinetic term in the form $\partial_\tau - \mu$ which in frequency momentum space takes the form $i(\omega_n + i\mu)$. This suggests that the fermion propagator at finite temperature is obtained by replacing the fourth component of a momentum as follows

$$p_4 \rightarrow \omega_n + i\mu \quad \omega_n = (2n+1)\pi/\beta \tag{76}$$

2 Polyakov loop and center symmetry

2.1 Introduction

In this section we want to study the static quark-antiquark potential at finite temperature. Static means, that the quarks are infinitely heavy and therefore their motion is frozen. In this section we will use lattice gauge theory.

At zero temperature the potential can be determined by studying the Wilson loop for large euclidean times. Since the lattice has a finite extension in the time direction, we cannot go to large euclidean times. To solve this problem we will consider the Polyakov loop instead.

[5] Here $p^2 = p_4^2 + \mathbf{p}^2$

2.2 Actions and gauge transformations on the lattice

The $SU(N)$ gauge action which describes the gauge bosons is given by

$$S_G = \frac{1}{2}\mathrm{Tr}\int \mathrm{d}^4x\, F_{\mu\nu}F_{\mu\nu} \tag{77}$$

where $F_{\mu\nu}$ is the field strength tensor defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{78}$$

where the matrices A_ν represent the gauge field and belong to the Lie algebra of $SU(N)$. If we want the fermionic action to be invariant under local $SU(N)$ gauge transformations $G(x)$ we should replace the ordinary four-derivative by the covariant derivative $D_\mu = \partial_\mu + \mathrm{i}gA_\mu$. The fermionic (Dirac) action $\int d^4x\bar{\psi}(\gamma_\mu D_\mu + M)\psi$ is then invariant under the following local transformations

$$\begin{aligned}\Psi(x) &\rightarrow G(x)\Psi(x)\\ \bar{\Psi}(x) &\rightarrow \bar{\Psi}(x)G^{-1}(x)\\ A_\mu(x) &\rightarrow G(x)A_\mu(x)G^{-1}(x) - \frac{\mathrm{i}}{g}G(x)\partial_\mu G^{-1}(x)\end{aligned} \tag{79}$$

The lattice is a division of space time into discrete points separated by the lattice constant a. They are called sites and can be labeled by a four vector $n = (n_1, n_2, n_3, n_4)$ with integer values (n_4 in euclidean time). On the lattice, it is necessary to have a gauge invariant expression for $\bar{\Psi}(x)\Psi(y)$ because we want to replace the covariant derivative by finite differences of the fields. The expression above is certainly not invariant under local $SU(N)$ transformations, but we can introduce the so called link variables given by

$$U_\mu(n) = \mathrm{P}e^{\mathrm{i}g\int_n^{n+\hat{\mu}} \mathrm{d}z_\nu A_\nu(z)} \tag{80}$$

where $\hat{\mu}$ is a vector of lenght a pointing along the μ direction and P denotes path ordering.[6] The link variables (or links) are again elements of $SU(N)$. Under a gauge transformation, the links transform as

$$U_\mu(n) \rightarrow G(n)U_\mu(n)G^{-1}(n+\hat{\mu}) \tag{81}$$

The gauge invariant expression replacing $\bar{\Psi}(n)\Psi(n+\hat{\mu})$ now is

$$\bar{\Psi}(n)U_\mu(n)\Psi(n+\hat{\mu}) \tag{82}$$

If we now construct any path on the lattice and consider the path ordered product of link variables, we see that it transforms as follows

$$U(n,m) = U(n,n+\hat{\mu})U(n+\hat{\mu},n+\hat{\mu}+\hat{\rho})\cdots U(n+\cdots,m) \rightarrow G(n)U(n,m)G^{-1}(m) \tag{83}$$

[6] The path ordering is introduced because $SU(N)$ is a non abelian group. It ensures that the link transforms according to (81).

We also want to have a gauge invariant expression for the gauge action on the lattice. In order to do so, we mention, that the trace of a closed loop is gauge invariant because of the cyclic property of the trace operation:

$$\mathrm{Tr}U(n,n) \to \mathrm{Tr}\left[G(n)U(n,n)G^{-1}(n)\right] = \mathrm{Tr}U(n,n) \tag{84}$$

We can express the gauge action (77) on the lattice by closed loops around so called plaquettes of the lattice. A plaquette in the $\mu-\nu$ plane are the four links forming a square with corners $n, n+\hat{\mu}, n+\hat{\mu}+\hat{\nu}, n+\hat{\nu}$. The plaquette variables are then

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n) \tag{85}$$

and it can be shown that the action (77) on the lattice is

$$S_G = \beta \sum_{n,\mu<\nu} [1 - \mathrm{Tr}(U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n))/2N] \tag{86}$$

which is gauge invariant because of (84). Here, $\beta = 2N/g^2$.

2.3 Center symmetry

The lattice action (also in continuum) of the SU(N) gauge theory is not only invariant under periodic gauge transformations. It also possesses the so called center symmetry. The center $\mathcal{C} \subset G$ of a group G consists of all elements $z \in G$ for which $zgz^{-1} = g \,\forall g \in G$ [7]. For SU(N) it is given by the identity matrix times $\exp(\frac{2\pi i l}{N})$, where $l = 0, 1 \cdots N-1$. Consider multiplying all time-like oriented link variables in a euclidean time slice, e.g. $n_4 = 0$ by an element of the center

$$U_4(\vec{n},0) \to zU_4(\vec{n},0) \tag{87}$$

Let us look at a plaquette

$$\begin{aligned} U_{i4}(\vec{n},0) &= U_i(\vec{n},0)U_4(\vec{n}+\hat{i},0)U_i^\dagger(\vec{n},1)U_4^\dagger(\vec{n},0) \\ &\to U_i(\vec{n},0)zU_4(\vec{n}+\hat{i},0)U_i^\dagger(\vec{n},1)U_4^\dagger(\vec{n},0)z^\dagger \end{aligned} \tag{88}$$

Multiplying the second and the fourth link by z does not change anything, because z commutes with all link variables and $zz^\dagger$ is the identity. Since the action is composed of plaquettes, it is invariant.

[7] This means that the elements of the center commute with all group elements

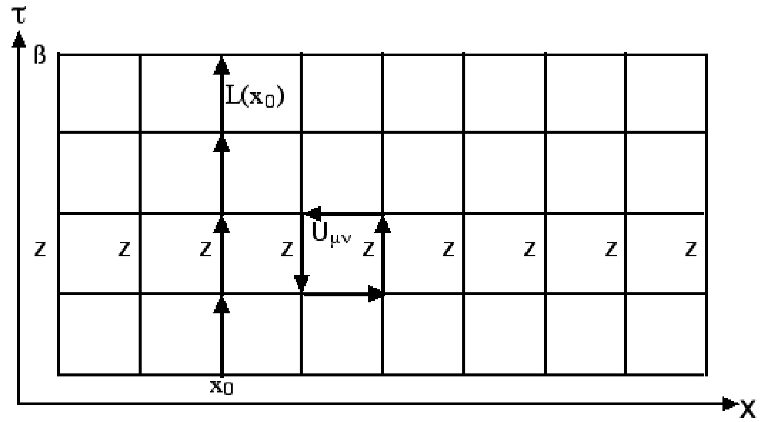

Figure 1: Example of a center transformation. The links pointing in time direction are multiplied by an element z of the center in a time slice. Here the slice is $n_4 = 1$. All plaquettes $U_{\mu\nu}$ with $\nu = 4$ are invariant under these transformations but the Polyakov loop is not.

2.4 The Polyakov loop

As mentioned before, the potential of a static quark-antiquark pair at zero temperature can be determined by examining the ground state expectation value of the Wilson loop for large euclidean times. At finite temperature, due to the finite extension of the lattice in the euclidean time direction, we have to consider another quantity, namely the Polyakov loop. It is the trace of the product of the link variables along a loop, winding around the euclidean time direction. Consider such a loop, located on a spatial lattice site $\vec{n}$

$$L(\vec{n}) = \frac{1}{N} \mathrm{Tr} \prod_{n_4=0}^{N_\tau - 1} U_4(\vec{n}, n_4) \tag{89}$$

Here $N_\tau = \beta/a$ is the number of lattice sites along the temporal direction. Clearly, this expression is invariant under periodic gauge transformations

$$\mathrm{Tr} \prod_{n_4=0}^{N_\tau-1} U_4(\vec{n}, n_4) \to \mathrm{Tr} \prod_{n_4=0}^{N_\tau-1} G(\vec{n}, n_4) U_4(\vec{n}, n_4) G^{-1}(\vec{n}, n_4+1) = \mathrm{Tr} \prod_{n_4=0}^{N_\tau-1} U_4(\vec{n}, n_4) \tag{90}$$

due to the cyclic property of the trace and $G(\vec{n}, 0) = G(\vec{n}, N_\tau)$. The Polyakov loop is a trace of a special unitary matrix. It can get values in the complex plane. The most peripheral values are plotted in fig. 2.

Since the Polyakov contains one link which transforms non-trivially under center transformations it is not invariant unless it is zero. This suggests that there exist two phases,

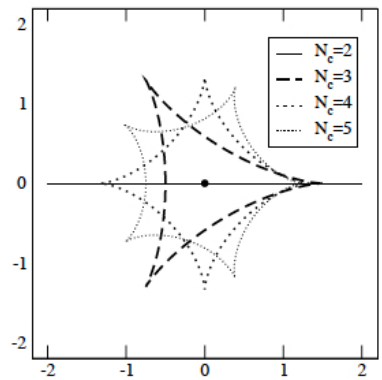

Figure 2: The most peripheral values of the Polyakov loop in the complex plane are plotted for N=2,3,4,5. If the system is in the deconfined phase the Polyakov loop takes values in the corners.

one which respects the center symmetry and the other which spontaneously breaks it. If the expectation value of the polyakov loop is zero, $\langle L \rangle = 0$, we say that the system is in the confined phase. We will see later why we have chosen that name. If $\langle L \rangle \neq 0$ we say that it is in the deconfined phase. If the center symmetry is spontaneously broken, there should be N distinct possible values for $\langle L \rangle$, with

$$\langle L \rangle = e^{2\pi \mathrm{i} l/N} L_0 \quad (l = 0, 1, \cdots, N-1) \tag{91}$$

Because the action is invariant, all these possible values occur with equal probability. Such expectation values minimize the action and the phase of the Polyakov loop clusters around any of these values. The expectation value therefore serves as an order parameter for distinguishing a confined from a deconfined phase. Numerical calculations show that there are indeed two phases separated by a phase transition. The confining phase is realized at low temperatures and the deconfining at high temperatures. The unbroken phase is disordered. At the critical temperature the system starts to form domains corresponding to the distinct values, the expectation value of the Polyakov loop can get. As the temperature is raised further the domains grow. Applying a "magnetic field" will flip the domains, resulting in a non vanishing expectation value for the Polyakov loop. It remains in this deconfining minimum after turning off the field again.

2.5 Physical meaning of the Polyakov loop

What is the physical meaning of the Polyakov loop? To answer this question we go back to the continuum formulation and consider a system consisting of an infinitely heavy quark coupled to a gauge potential with a Hamiltonian H. For simplicity we consider the $U(1)$

gauge group. Let $|s\rangle$ denote the states containing the heavy quark, and $|s'\rangle$ the states which do not. Then the partition function of the infinitely heavy quark in a heat bath of gluons is given by:

$$Z = \sum_s \langle s|e^{-\beta H}|s\rangle \tag{92}$$

We can now create the quark located at $\vec{x}$ and time $x_4 = 0$ by applying creation operators to the states $|s'\rangle$. The partition function now is[8]

$$\begin{aligned} Z &= N\sum_{s'}\langle s'|\Psi(\vec{x},0)e^{-\beta H}\Psi^\dagger(\vec{x},0)|s'\rangle \\ &= N\sum_{s'}\langle s'|e^{-\beta H}\Psi(\vec{x},\beta)\Psi^\dagger(\vec{x},0)|s'\rangle \end{aligned} \tag{93}$$

The factor N takes care of the normalization of the quark state. The time evolution of Ψ is given by the Dirac equation. Since we are dealing with an infinitely heavy quark, the spatial derivatives can be neglected[9]. The evolution therefore is given by (after multiplying by γ_4):

$$(\partial_\tau - \mathrm{i}eA_4(\vec{x},\tau))\Psi(\vec{x},\tau) = 0 \tag{94}$$

This equation has a formal solution

$$\Psi(\vec{x},\beta) = e^{\mathrm{i}e\int_0^\beta \mathrm{d}\tau A_4(\vec{x},\tau)}\Psi(\vec{x},0) = L(\vec{x})\Psi(\vec{x},0) \tag{95}$$

where $L(\vec{x}) = e^{\mathrm{i}e\int_0^\beta \mathrm{d}\tau A_4(\vec{x},\tau)}$ is the Polyakov loop for $U(1)$[10]. Inserting this into (93) yields

$$Z = N\sum_{s'}\langle s'|e^{-\beta H}L(\vec{x})\Psi(\vec{x},0)\Psi^\dagger(\vec{x},0)|s'\rangle \tag{96}$$

The operator $\Psi(\vec{x},0)\Psi^\dagger(\vec{x},0)$ only gives rise to an (infinite) constant, which is cancelled by the normalization factor. Intuitively, inserting $L(\vec{x})$ into the expression for the partition function $\mathrm{Tr}e^{-\beta \mathrm{H}}$ of the pure gauge theory, corresponds to including a single static quark. A way to interpret the Polyakov loop is to relate it to the free energy of the system. For that purpose, we go back to Eq. (96) and use the thermodynamic relation

$$Z = e^{-\beta F} \tag{97}$$

which gives

$$e^{-\beta F_q} = \mathrm{Tr}[\exp(-\beta H)L(\vec{x})] \tag{98}$$

[8] $e^{\beta H}\Psi(\vec{x},0)e^{-\beta H} = \Psi(\vec{x},\beta)$

[9] We dropped the mass term since it only gives rise to an extra phase which is cancelled.

[10] The SU(N) case follows by path ordering. $L(\vec{x}) = Pe^{ie\int_0^\beta \mathrm{d}\tau A_4(\vec{x},\tau)}$

This is the free energy of the entire system of gluons plus quark. To obtain the free energy difference ΔF_q between the system with quark and without quark, we have to divide the expression by the partition function of the pure gauge theory (this means with gluons only)

$$Z = \mathrm{Tr} e^{-\beta H} \tag{99}$$

and we get the expectation value of the Polyakov loop

$$e^{-\beta \Delta F_q} = \langle L \rangle \tag{100}$$

We immediately see that if the expectation value of the Polyakov loop vanishes, the free energy of the quark is infinite. If the quark is very heavy but finitely heavy the expectation value is forced onto the real axis.
We can also consider creating a quark and an antiquark at some different spatial points $\vec{x}$ and $\vec{x'}$. The free energy of this system, obtained in a similar way, reads[11]

$$e^{-\beta \Delta F_{q\bar{q}}} = \langle L(\vec{x}) L^\dagger(\vec{x'}) \rangle \tag{101}$$

One can show that

$$\langle L(\vec{x}) L^\dagger(\vec{x'}) \rangle \to |\langle L \rangle|^2 \qquad (|\vec{x} - \vec{x'}| \to \infty) \tag{102}$$

which also means that if L vanishes, we have an increasing free energy with increasing separation of the quarks. This signals confinement.
On the other hand, if $\langle L \rangle \neq 0$, the free energy approaches a finite constant for large separations. We interpret this as signaling deconfinement. But this is what we expect. The coupling gets weaker and weaker as the temperature is raised (asymptotic freedom), hence deconfinement is plausible at high temperatures.

Let us look at a system containing N_q quarks and $N_{\bar{q}}$ antiquarks. Then the generalization of (101) is

$$e^{-\beta \Delta F_{N_q N_{\bar{q}}}} = \langle L(\vec{x}_1) \cdots L(\vec{x}_{N_q}) L^\dagger(\vec{x}'_1) \cdots L^\dagger(\vec{x}'_{N_{\bar{q}}}) \rangle \tag{103}$$

which transforms as follows under center transformations:

$$e^{-\beta \Delta F_{N_q N_{\bar{q}}}} \to e^{-\beta \Delta F_{N_q N_{\bar{q}}}} e^{2\pi \mathrm{i} l (N_q - N_{\bar{q}})/N} \tag{104}$$

In the confined phase the center symmetry is not broken and the expression above should not change, too. This means that, unless $N_q - N_{\bar{q}}$ is a multiple integer of N, the free energy of an assembly of quarks and antiquarks must be infinite. For $SU(3)$ it is therefore only possible to form assemblies if the number of quarks minus the number of antiquarks is an integer multiple of 3.

[11] Because we consider an antiquark we have to consider a Polyakov loop oriented in the opposite direction, traveling backwards in euclidean time. That is why we put a dagger.

2.6 Some simulations

In order to do simulations on computers, one must choose appropriate temporal and spatial extensions of the lattice. The simulation should be insensitive to the finite spatial extension of the lattice. If we would choose a lattice which has the same number of lattice sites along the temporal direction as along the spatial, physics would also be insensitive to periodic boundary conditions (temperature). Therefore, the temporal extension must be much smaller than the spatial. The temperature is given by

$$T = \frac{1}{N_\tau a} \tag{105}$$

where a is the lattice spacing and N_τ is the number of lattice sites along the temporal direction. It can be shown that the lattice spacing is related to the coupling β (which appears in the lattice gauge action eqn. (86)). Hence, it is possible to vary the temperature, by varying the coupling. The following simulations show evidence that $SU(3)$ gauge theories exhibit a phase transition. Fig. 3. shows the expectation value of the Polyakov loop as

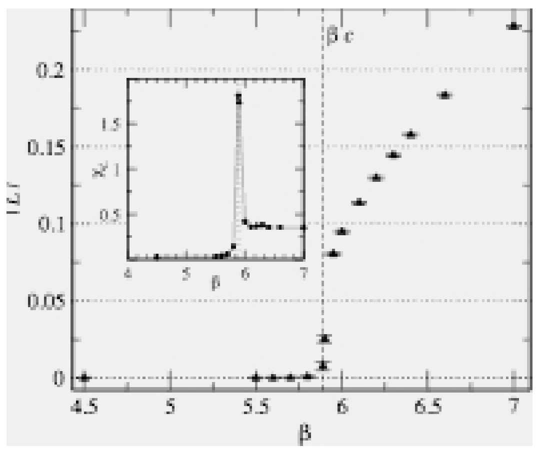

Figure 3: Absolute value and susceptibility of the Polyakov loop as a function of the coupling. This simulation has been performed on a $16^3 \times 6$ lattice. Figure taken from [3].

a function of the coupling β. This simulation has been performed on a $16^3 \times 6$ lattice. It is clearly visible, that at low temperatures (which correspond to small couplings) the Polyakov loop is almost zero signaling confinement. Above the critical coupling the order parameter has increased to a finite value. The system is in the deconfined phase. The inset shows the susceptibility of the Polyakov loop which is its variance. Fig. 5. shows the distribution of real and imaginary parts of Polyakov loops for different couplings. In the deconfined center symmetric phase, the points are distributed around the origin.
Near the critical coupling the distribution broadens. It does not cluster around a specific

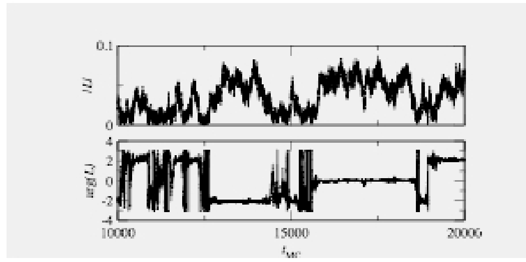

Figure 4: Absolute value and phase of the Polyakov loop as a function of simulation time near the critical coupling. Horizontal lines show that the system is in the deconfined phase. This simulation has been performed on a $16^3 \times 6$ lattice. Figure taken from [3].

value because the Polyakov loop tunnels from one value to another during the simulation, showing coexistence of the two phases. This is shown in Fig. 4. It shows the phase and the absolute value of the Polyakov loop as a function of time during the simulation performed near the critical coupling. The horizontal lines show that the phase of the Polyakov loop resides at one of the three values. Thus we are in the deconfined phase. There, the absolute value does not vanish. In the remaining time intervals the system is in the unbroken confined phase and the phase angle changes wildly.
At higher couplings, the Polyakov loop clusters around a non-vanishing value, which means that the system is in the center symmetry broken deconfined phase.

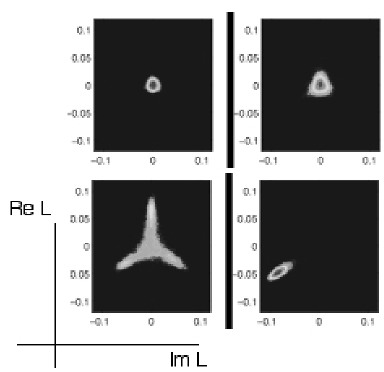

Figure 5: Distribution of the real and imaginary part of the Polyakov loop. Top left and right: confined phase at lower couplings. The data is distributed around the origin. Bottom left: The distribution broadens near the critical coupling showing coexistence of the phases. Bottom right: Above the critical coupling the Polyakov loop clusters around one value of the center and is in the deconfined phase. Figure taken from [3].

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