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Perturbative and Non-Perturbative Methods for Strong Interactions

Goldstone's Theorem and Chiral Symmetry Breaking

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1 Introduction

This talk aims at introducing the main concepts that can be derived from symmetry properties of a physical theory.
In the first part we reconsider the definition of symmetry. We then derive Noether's theorem in a classical theory, which relates symmetries and conserved quantities, the so called Noether currents and Noether charges. We then transfer our results from classical mechanics to quantum mechanics.
The results are applied to QCD where we discuss chiral symmetry in the massless limit. We introduce vector and axial currents.
The second part of the talk deals with spontaneous symmetry breaking. We first discuss spontaneous symmetry breaking in the classical theory where it has a very intuitive meaning as a shift of the origin of the fields to the minimum of the potential. We already see the emergence of massless particles in the process - the Goldstone bosons. This will facilitate the transfer to quantum mechanics, where we will get a new interpretation of spontaneous symmetry breaking: It occurs when the vacuum is charged. We prove Goldstone's theorem which predicts the emergence of a massless scalar boson - a so called Goldstone boson - for each broken charge.
The results are again applied to QCD. We motivate why we expect the chiral symmetry to be spontaneously broken and we identify the emerging Goldstone bosons with the pions.

2 Symmetries and Conservation Laws

2.1 Noether's theorem in classical field theory

Consider a classical field theory, described by an n-component, real field ϕ, which is governed by Lagrangian dynamics, i.e. there is a Lagrange density

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \tag{1}$$

from which that the equations of motion can be obtained using the principle of minimal action. This process eventually yields the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0 \qquad i = 1, \cdots, n \tag{2}$$

For later purpose, we introduce the momentum variables canonically conjugate to the field variables

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i(x))}. \tag{3}$$

Now consider transformations of the fields. We will restrict ourselves to transformations under the fundamental representation of a matrix Lie group, i.e. which are (1) linear in the fields and (2) smoothly parametrised by k independent, real parameters $\theta_1, \cdots, \theta_k$ satisfying $R(\theta = 0) = id$:

$$\phi(x) \to \tilde{\phi}(x) = (R(\theta_1, \cdots, \theta_k))\phi(x) \tag{4}$$

We can express the elements of the transformation group in terms of basis vectors $\{\lambda_a \quad a = 1, \cdots, k\}$ of the Lie algebra using the exponential map:

$$R(\theta_1, \cdots, \theta_k) = e^{-i\theta_a \lambda_a} = 1 - i\theta_a \lambda_a + 0(\theta^2). \tag{5}$$

The basis vectors λ_a of the Lie algebra satisfy the commutation relations

$$[\lambda_a, \lambda_b] = i f_{abc} \lambda_c. \tag{6}$$

where the complex numbers f_{abc} are called the structure constants.
If we are only interested in the behaviour under transformations close to the identity transformation, we can do a Taylor expansion to linear order:

$$\phi(x) \to \tilde{\phi}(x) = \phi(x) - i\theta_a \lambda_a \phi(x). \tag{7}$$

Having completed these elementary remarks, we can now go on to discuss the variation of the Lagrange density under a transformation of the fields. We find:

$$\delta\mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \mathcal{L}(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \theta_a \partial_\mu \left(-i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \lambda_a \phi \right) \tag{8}$$

Having come so far, we easily arrive at Noether's theorem. Assume, that there is a group of global transformations (i.e. transformations that are space-time independent, $\theta_a \neq \theta_a(x)$) which do not change the Lagrange density (i.e. $\delta\mathcal{L} = 0$). We then find:

$$0 = \delta\mathcal{L} = \theta_a \partial_\mu \left(-i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \lambda_a \phi \right) \tag{9}$$

Since the θ-parameters are all independent, we find k Noether-currents

$$J_a^\mu(x) = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \lambda_a \phi(x) \tag{10}$$

that satisfy the continuity equation

$$\partial_\mu J_a^\mu = 0. \tag{11}$$

To each Noether current we have an associated charge

$$Q_a(t) = \int J_a^0(x) \mathrm{d}^3x = -i \int \pi(x) \lambda_a \phi(x) \mathrm{d}^3x \tag{12}$$

Using the continuity equation and Gauss' law, we can easily convince ourselves that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_a(t) = \int \frac{\partial J_a^0(x)}{\partial t} \mathrm{d}^3x = -\int div \vec{J}_a(x) \mathrm{d}^3x = 0. \tag{13}$$

The charges Q_a are therefore called conserved Noether charges.

2.2 Symmetries in a quantized theory

In the process of canonical quantization, the fields $\phi_i(x)$ and the momenta $\pi_i(x)$ are promoted to linear operators $\hat{\phi}_i(x)$ and $\hat{\pi}_i(x)$ acting on a Hilbert space $\mathcal{H}$ and satisfying canonical equal-time commutation relations.

$$\begin{aligned} [\hat{\phi}_i(t,\vec{x}), \hat{\phi}_j(t,\vec{y})] = [\hat{\pi}_i(t,\vec{x}), \hat{\pi}_j(t,\vec{y})] = 0 \\ [\hat{\phi}_i(t,\vec{x}), \hat{\phi}_j(t,\vec{y})] = i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{14}$$

By analogy to the classical case, we can define charge and current operators acting on the underlying Hilbert space by replacing the field and momentum variables in equations 10 and 12 by the corresponding field variables, i.e.

$$\begin{aligned} J_a^\mu(x) &= -i\tfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}(\hat{\phi}(x), \hat{\pi}(x))\lambda_a\hat{\phi}(x) \\ \hat{Q}_a(t) &= -i\textstyle\int \hat{\pi}(t,\vec{x})\lambda_a\hat{\phi}(t,\vec{x})\mathrm{d}^3x \end{aligned} \tag{15}$$

where the λ_a span the Lie algebra of some group describing field transformations.

Computing the commutators between charge operators, we find an interesting new feature of quantum theory:

$$[\hat{Q}_a(t), \hat{Q}_b(t)] = i\int \hat{\pi}(t,\vec{x})[\lambda_a,\lambda_b]\hat{\phi}(t,\vec{x})\mathrm{d}^3x = if_{abc}\hat{Q}_c(t) \tag{16}$$

i.e. the charges form a Lie algebra with the same structure constants as the Lie algebra of the underlying transformation group. This implies in particular that the charges can take quantized values only.

Next, let us compute the commutators between the charge operators and the field operator. We easily find:

$$[Q_a(t), \phi_i(t,\vec{x})] = \int [\hat{\pi}(t,\vec{y})\lambda_a\hat{\phi}(t,\vec{y}), \phi_i(t,\vec{x})]\mathrm{d}^3y = -\lambda_a\phi(x) \tag{17}$$

This relation is helpful to consider the action of a field transformation on the underlying Hilbert space. Let $|\alpha\rangle$ be an element of this Hilbert space. Each transformation

$$\phi \to \tilde{\phi} = R(\theta)\phi \tag{18}$$

of the field operators will of course also affect the states, mapping

$$|\alpha\rangle \to |\tilde{\alpha}\rangle = U(\theta)\,|\alpha\rangle\,. \tag{19}$$

Requiring that

$$\tilde{\phi}\,|\tilde{\alpha}\rangle = R(\theta)\phi U(\theta)\,|\alpha\rangle = U(\theta)(\phi\,|\alpha\rangle) = (\phi\,\widetilde{|\alpha\rangle}) \tag{20}$$

we find the remarkable result

$$U(\theta) = e^{i\theta^a Q_a}. \tag{21}$$

We have thus found an interpretation of the charge operators as being the infinitesimal generators of the representation of the transformation Lie group

on the underlying Hilbert space.
Note that the previous definitions and relations hold for completely arbitrary transformation groups. There is of course a subgroup of particular importance, namely those transformations which are symmetries of the Lagrangian in the sense discussed in the previous section.In this case, the current operators satisfy the continuity equation

$$\partial_\mu \hat{j}^\mu_a(x) = 0 \tag{22}$$

and the corresponding charge operators are time independent:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{Q}_a(t) = 0 \quad \Rightarrow \quad [\hat{Q}_a, \hat{H}] = 0 \tag{23}$$

where $\hat{H}$ is the Hamilton operator of the quantum field theory.

3 Chiral Symmetry of QCD

3.1 The QCD Lagrangian

In Quantum Chromodynamics (QCD) we have six different quarks: up, down, strange, charm, top and bottom. They are commonly called quark flavours. The masses of the six quarks are strikingly different, thus it often makes sense to divide them into the light quark sector and the heavy quark sector

$$\begin{pmatrix} m_u = 0.005 \text{ GeV} \\ m_d = 0.009 \text{ GeV} \\ m_s = 0.175 \text{ GeV} \end{pmatrix} \quad \ll \quad 1 \text{ GeV} \quad < \quad \begin{pmatrix} m_c \approx 1.2 \text{ GeV} \\ m_b \approx 4.2 \text{ GeV} \\ m_t \approx 174 \text{ GeV} \end{pmatrix} \tag{24}$$

Additionally, each quark flavour appears in three different colours (red, green, blue) such that the full quark wavefunctions read

$$q_f = \begin{pmatrix} q_{f,r} \\ q_{f,g} \\ q_{f,b} \end{pmatrix}. \tag{25}$$

Note that $q_{f,c}$ are Dirac 4-spinor valued fields on Minkowski space.
The Lagrange function of QCD is given by

$$\mathcal{L}_{QCD} = \sum_f \overline{q}_f (i\gamma^\mu \mathrm{D}_\mu - m_f) q_f - \frac{1}{4} \mathcal{G}_{a,\mu\nu} \mathcal{G}_a^{\mu\nu}. \tag{26}$$

where

$$\mathrm{D}_\mu = \partial_\mu - igA_\mu \qquad A_\mu = A_{a,\mu} \frac{\lambda_a^C}{2} \tag{27}$$

is the covariant derivative. The matrices λ_a^C, a=1,...,8 are the Gell-Mann matrices - i.e. twice the generators of SU(3), the Lie group of the underlying colour gauge symmetry.
The dynamics of the gauge fields is governed by the field strength tensor

$$\mathcal{G}_{\mu\nu} = \mathcal{G}_{a,\mu\nu} \frac{\lambda_a^C}{2} = \frac{1}{ig} [D_\mu, D_\nu] \qquad \mathcal{G}_{a,\mu\nu} = \partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu} + g f_{abc} A_{b,\mu} A_{c,\nu} \tag{28}$$

For many applications in low energy QCD, we can ignore the quantum corrections that come from virtual quark-antiquark pairs $h\overline{h}$ of the heavy quarks. We may thus restrict ourselves to a reduced Lagrange function that contains only the light up, down and strange quarks:

$$\mathcal{L} = \sum_{f=u,d,s} \overline{q}_f (i\gamma^\mu \mathrm{D}_\mu - m_f) q_f - \frac{1}{4} \mathcal{G}_{a,\mu\nu} \mathcal{G}_a^{\mu\nu}. \tag{29}$$

3.2 Colour Symmetry

QCD has been constructed as a gauge field theory with an underlying exact $SU(3)_{colour}$ gauge symmetry, meaning that the Lagrangian is invariant under local gauge transformations

$$q_f(x) \to G(x)q_f(x) \qquad A_\mu(x) \to G(x)A_\mu(x)G^\dagger(x) + \frac{i}{g}(\partial_\mu G(x))G^\dagger(x) \tag{30}$$

where $G(x) \in SU(3)$. We are however not going to discuss this symmetry.

3.3 Chiral Symmetry

Before we can continue with discussing chiral symmetry, we first have to clarify, what chirality is. To that end, introduce the chirality matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{31}$$

satisfying the following very important properties

$$\gamma^5 = \gamma_5 = \gamma_5^\dagger \quad (\gamma^5)^2 = 1 \quad \{\gamma^5, \gamma^\mu\} = 0 \tag{32}$$

Using the chirality matrix, we can next define the projection operators to the left- and right-handed components of a Dirac-field

$$P_L = \frac{1}{2}(1-\gamma^5) \qquad P_R = \frac{1}{2}(1+\gamma^5) \tag{33}$$

These operators indeed have the idempotence property

$$P_L^2 = P_L \qquad P_R^2 = P_R \tag{34}$$

and further satisfy completeness and orthogonality relations

$$P_L + P_R = 1 \qquad P_L P_R = P_R P_L = 0. \tag{35}$$

From the first equation in 35 we see that every Dirac field can be expanded in terms of its left- and right-handed components. We can therefore rewrite the Lagrange density of the light-quark sector as

$$\begin{aligned} \mathcal{L} &= \textstyle\sum_{f=u,d,s} \overline{q}_f(i\gamma^\mu \mathrm{D}\mu - m_f)q_f - \frac{1}{4}\mathcal{G}_{a,\mu\nu}\mathcal{G}_a^{\mu\nu} \\ &= \textstyle\sum_{f=u,d,s}\{\overline{q}_{f,L}(i\gamma^\mu \mathrm{D}_\mu)q_{f,L} + \overline{q}_{f,R}(i\gamma^\mu \mathrm{D}_\mu)q_{f,R} \\ &\quad -m_f\overline{q}_{f,L}q_{f,R} - m_f\overline{q}_{f,R}q_{f,L}\} - \frac{1}{4}\mathcal{G}_{a,\mu\nu}\mathcal{G}_a^{\mu\nu} \end{aligned} \tag{36}$$

where the left- and right-handed components are shown explicitly. In the course of the calculation, we have made use of

$$\begin{aligned} &\overline{q}_{f,L}\gamma^0(i\gamma^\mu \mathrm{D}_\mu)q_{f,R} = q_f^\dagger P_L^\dagger \gamma^0(i\gamma^\mu \mathrm{D}_\mu)P_R q_f = \overline{q}_f(i\gamma^\mu \mathrm{D}_\mu)P_L P_R q_f = 0 \\ &\overline{q}_{f,L}\gamma^0(i\gamma^\mu \mathrm{D}_\mu)q_{f,R} = \cdots = 0. \end{aligned} \tag{37}$$

3.3.1 Phase Invariance of QCD with Massless Quarks

Let us in a first step review the Lagrangian of QCD in the light quark sector and set all masses to zero. The Lagrange density in this limit reads

$$\mathcal{L}^0 = \sum_{f=u,d,s} \{\overline{q}_{f,L}(i\gamma^\mu \mathrm{D}_\mu)q_{f,L} + \overline{q}_{f,R}(i\gamma^\mu \mathrm{D}_\mu)q_{f,R}\} - \frac{1}{4}\mathcal{G}_{a,\mu\nu}\mathcal{G}_a^{\mu\nu}. \tag{38}$$

This Lagrangian describes fields with independent left- and right-handed components.
We easily verify, that it is invariant under global phase shifts

$$q_{f,L} \to e^{-i\theta_L} q_{f,L} \qquad q_{f,R} \to e^{-i\theta_R} q_{f,R} \tag{39}$$

forming the symmetry group $U(1)_L \times U(1)_R$. For future convenience, we write these transformations in a different form. Let us define vector and axial transformations by

$$\text{Vec}: \begin{pmatrix} q_{f,L} \\ q_{f,R} \end{pmatrix} \to e^{-i\theta_V} \begin{pmatrix} q_{f,L} \\ q_{f,R} \end{pmatrix} \qquad \text{Ax}: \begin{pmatrix} q_{f,L} \\ q_{f,R} \end{pmatrix} \to \begin{pmatrix} e^{i\theta_A} q_{f,L} \\ e^{-i\theta_A} q_{f,R} \end{pmatrix} \tag{40}$$

Obviously equations (39) and (40) are equivalent in the sense that they describe the same group of transformations

$$U(1)_L \times U(1)_R = U(1)_V \times U(1)_A. \tag{41}$$

Further noting that

$$e^{i\theta_A} P_L + e^{-i\theta_A} P_R = \sum_{k=0}^{\infty} \frac{(i\theta_A)^k}{k!} \underbrace{\{P_L + (-1)^k P_R\}}_{=(-\gamma^5)^k} = e^{-i\theta_A \gamma^5} \tag{42}$$

We can rewrite the symmetry transformations in equation (40) in a very elegant way:

$$\text{Vec}: q_f \to e^{-i\theta_V} q_f \qquad \text{Ax}: q_f \to e^{-i\theta_A \gamma^5} q_f \tag{43}$$

The generators of the Lie-groups $U(1)_V$ and $U(1)_A$ are given by

$$\lambda_V = 1 \qquad \lambda_A = \gamma^5. \tag{44}$$

We are now really in a position to apply Noether's theorem to obtain the following conserved Noether currents V and A associated to the vector and axial symmetry transformations.

$$V^\mu(x) = -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_f(x))}\lambda_V q_f(x) = \overline{q}_f(x)\gamma^\mu q_f(x) \tag{45}$$

$$A^\mu(x) = -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_f(x))}\lambda_A q_f(x) = \overline{q}_f(x)\gamma^\mu \gamma^5 q_f(x) \tag{46}$$

3.3.2 Chiral Symmetry of QCD with massless quarks

The symmetry with respect to phase shifts is not everything we have. In fact - noting that the covariant derivative is independent of flavour - we immediately see that rotating the left- and right-handed components independently in flavour space

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \\ s_{L,R} \end{pmatrix} \to F_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \\ s_{L,R} \end{pmatrix} = e^{-i\theta^b_{L,R}\frac{\lambda^F_b}{2}} \begin{pmatrix} u_{L,R} \\ d_{L,R} \\ s_{L,R} \end{pmatrix} \tag{47}$$

with $F_{L,R} \in SU(3)$ also leaves the Lagrangian invariant (the superscript F indicates, that the Gell-Mann λ-matrices act in flavour space). These symmetries form the *chiral symmetry group* $SU(3)_L \times SU(3)_R$. Performing the same steps as in the previous section, we can replace

$$SU(3)_L \times SU(3)_R \quad \to \quad SU(3)_V \times SU(3)_A \tag{48}$$

and the vector and axial transformations are defined as

$$\text{Vec}: \begin{pmatrix} u \\ d \\ s \end{pmatrix} \to e^{-i\theta^b_V \frac{\lambda^F_b}{2}} \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad \text{Ax}: \begin{pmatrix} u \\ d \\ s \end{pmatrix} \to e^{-i\theta^b_A \frac{\lambda^F_b}{2}\gamma^5} \begin{pmatrix} u \\ d \\ s \end{pmatrix}. \tag{49}$$

The generators of these symmetry transformations given by

$$\lambda_{V,b} = \frac{\lambda^F_b}{2} \qquad \lambda_{A,b} = \frac{\lambda^F_b}{2}\gamma^5 \tag{50}$$

we can directly write down the corresponding conserved Noether currents V_b and A_b

$$V^\mu_b(x) = \overline{q}(x)\gamma^\mu \frac{\lambda^F_b}{2} q(x) \tag{51}$$

$$A^\mu_b(x) = \overline{q}(x)\gamma^\mu\gamma^5 \frac{\lambda^F_b}{2} q(x) \tag{52}$$

The total symmetry group in the massless limit is $SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$.

3.4 Chiral Symmetry Breaking by Quark Masses

Recall that we have only considered the extremely simplified Lagrangian $\mathcal{L}^0$ in (38) where we have ignored the heavy quarks and set the remaining masses to zero. What happens, if we now put the mass terms

$$\begin{aligned} \mathcal{L}^M &= (\overline{q}_L + \overline{q}_R) M (q_f + q_R) \\ &= \begin{pmatrix} \overline{u}_L + \overline{u}_R & \overline{d}_L + \overline{d}_R & \overline{s}_L + \overline{s}_R \end{pmatrix} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \begin{pmatrix} u_L + u_R \\ d_L + d_R \\ s_L + s_R \end{pmatrix} \end{aligned} \tag{53}$$

back in such that $\mathcal{L} = \mathcal{L}^0 - \mathcal{L}^M$?
This mass term will not in general be invariant under the transformations defined in (40) and (49) and neither will the Lagrangian $\mathcal{L}$ be. The divergences of the Noether currents that are associated with the symmetries of $\mathcal{L}^0$ will then obtain a non-vanishing contribution from the mass terms.
We will carry out the computation in full detail for the Noether current A_b^μ. The computations for the other currents are completely analogous.

The Noether current

$$A_b^\mu(x) = \overline{q}(x)\gamma^\mu\gamma^5\frac{\lambda_b}{2}q(x) \tag{54}$$

has been derived from the transformation

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to e^{-i\theta\lambda_b^F\gamma^5}\begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{55}$$

with infinitesimal generator

$$\lambda_{A,b} = \frac{\lambda_b^F}{2}\gamma^5 \tag{56}$$

From (8), we see that the divergences of the currents defined in (10) are exactly the first order variations of the Lagrangian, i.e.

$$\begin{aligned}
\partial_\mu V_b^\mu &= \frac{\partial(\delta\mathcal{L})}{\partial\theta}\mid_{\theta=0} = \frac{\partial}{\partial\theta}\underbrace{\delta\mathcal{L}^0}_{=0}\mid_{\theta=0} - \frac{\partial}{\partial\theta}\delta\mathcal{L}^M\mid_{\theta=0} \\
&= -\frac{\partial}{\partial\theta}\{\overline{q}\gamma^0\left(1+i\theta\lambda_{V,b}\right)\gamma^0 M\left(1-i\theta\lambda_{V,b}\right) - \overline{q}Mq\}\mid_{\theta=0} \\
&= -i\overline{q}\gamma^0[\lambda_{V,b},\gamma^0 M]q = i\overline{q}\gamma^0[M\gamma^0,\frac{\lambda_b^F}{2}\gamma^5] \\
&= i\overline{q}\{M,\frac{\lambda_b^F}{2}\}\gamma^5 q
\end{aligned} \quad . \tag{57}$$

We obtain the following divergences for the vector and axial currents:

$$\partial_\mu V^\mu = 0 \tag{58}$$
$$\partial_\mu A^\mu = 2i\overline{q}M\gamma^5 q \tag{59}$$
$$\partial_\mu V_b^\mu = i\overline{q}[M,\frac{\lambda_b^F}{2}]q \tag{60}$$
$$\partial_\mu A_b^\mu = i\overline{q}\{M,\frac{\lambda_b^F}{2}\}\gamma^5 q \tag{61}$$

Note that λ_3 and λ_8 commute with diagonal matrices. Thus there are three conserved currents that survive

$$\begin{array}{rclcl}
V^\mu &=& \overline{q}\gamma^\mu q &=& \overline{u}\gamma^\mu u + \overline{d}\gamma^\mu d + \overline{s}\gamma^\mu s \\
V_3^\mu &=& \overline{q}\gamma^\mu\frac{\lambda_3^F}{2}q &=& \frac{1}{2}\{\overline{u}\gamma^\mu u - \overline{d}\gamma^\mu d\} \\
V_8^\mu &=& \overline{q}\gamma^\mu\frac{\lambda_8^F}{2}q &=& \frac{1}{2\sqrt{3}}\{\overline{u}\gamma^\mu u + \overline{d}\gamma^\mu d\ - 2\overline{s}\gamma^\mu s\}
\end{array} \tag{62}$$

The sum of conserved currents being a conserved current itself, we use linear combinations of (62) to rewrite the conserved currents in a very intuitive way:

$$\begin{aligned} V_u^\mu &= \overline{u}\gamma^\mu u \\ V_d^\mu &= \overline{d}\gamma^\mu d \\ V_s^\mu &= \overline{s}\gamma^\mu s \end{aligned} \tag{63}$$

Note that if all quark masses were equal, i.e. if the mass matrix was proportional to the identity matrix, all the vector currents V_b^μ would remain to be conserved. This is the origin of the $SU(3)_{flavour}$ symmetry of strong interactions proposed by Gell-Mann and Ne'eman.
We have seen that the mass term does not have the full chiral symmetry of the kinetic term. The chiral symmetry is therefore called an *approximate symmetry*. It becomes exact in the massless limit (or in the limit of high energies accordingly). As the symmetry is broken by terms which are part of the Lagrangian, it is called *explicitly broken*.

3.5 Summary

Let us summarize what we have learned about chiral symmetry:

1. Ignoring the quark masses, the Lagrangian has a $SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$ symmetry in the fields. This is the joint symmetry group of chiral symmetry and phase invariance. Using Noether's Theorem we can construct 8+8+1+1=18 currents $V_b^\mu, A_b^\mu, V^\mu, A^\mu$ which are conserved in the massless limit.

2. The chiral symmetry group $SU(3)_V \times SU(3)_A$ is *explicitly broken* to $U(1)_u \times U(1)_d \times U(1)_s$ by the mass terms in the Lagrangian. Only the individual flavour currents $\overline{u}\gamma^\mu u, \overline{d}\gamma^\mu d, \overline{s}\gamma^\mu s$ remain to be conserved. This is a consequence of the flavour independence of the covariant derivative and the diagonality of the mass matrix, i.e. the Lagrangian is a sum of the contributions from different flavours with no terms that contain quarks with different flavours.

3. The currents associated to explicitly broken symmetries obtain a non-vanishing divergence. This divergence essentially is the variation of the Lagrange function under the transformation in question.

4 Spontaneous Symmetry Breaking and Goldstone's Theorem

4.1 Classical Version of Goldstone's Theorem

4.1.1 An Instructive Example

As a warm-up, let us consider a field theory with a two-component real field $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2)$ which is described by a Lagrange density

$$\mathcal{L}(\partial_\mu \tilde{\phi}, \tilde{\phi}) = \frac{1}{2}\partial_\nu \tilde{\phi}_1 \partial^\nu \tilde{\phi}_1 + \frac{1}{2}\partial_\nu \tilde{\phi}_2 \partial^\nu \tilde{\phi}_2 - \frac{\mu^2}{2}(\tilde{\phi}_1^2 + \tilde{\phi}_2^2) - \frac{\lambda}{4!}(\tilde{\phi}_1^2 + \tilde{\phi}_2^2)^2 \tag{64}$$

Apparently, this Lagrangian has a SO(2) symmetry of the fields

$$\tilde{\phi}(x) \to R(\theta)\tilde{\phi}(x) \qquad R(\theta) \in SO(2). \tag{65}$$

For $\mu^2 > 0$ this Lagrangian has a simple interpretation as describing two fields with mass μ each and interactions described by a coupling constant λ. This interpretation obviously fails for $\mu^2 < 0$ in which case we would have to deal with a negative mass. Looking at the full potential term

$$V(\phi) = \frac{\mu^2}{2}\tilde{\phi}_1(x)^2 + \frac{\mu^2}{2}\tilde{\phi}_2(x)^2 + \frac{\lambda}{4!}(\tilde{\phi}_1(x)^2 + \tilde{\phi}_2(x)^2)^2 \tag{66}$$

we further note that the minimum of the potential energy has been shifted from $\tilde{\phi}(x) = 0$ to a ring of minima at $|\tilde{\phi}(x)| = v = \sqrt{-\frac{6\mu^2}{\lambda}}$. It is much more natural to think of the fields as perturbations around the minimum of the potential energy term. Writing

$$\tilde{\phi}(x) = \begin{pmatrix} v + \phi_1(x) \\ \phi_2(x) \end{pmatrix} \tag{67}$$

we find the Lagrangian expressed in terms of the physical fields ϕ_1 and ϕ_2:

$$\begin{aligned} \mathcal{L}(\partial_\nu\phi_1(x), \partial_\nu\phi_2(x), \phi_1(x), \phi_2(x)) &= \left(\partial_\nu\phi_1(x)\partial^\nu\phi_1(x) + \tfrac{3\mu^2}{2}\phi_1(x)^2\right) \\ &+ \ (\partial_\nu\phi_2(x)\partial^\nu\phi_2(x)) \\ &+ \ (\text{cubic} + \text{quartic}) \end{aligned} \tag{68}$$

The interpretation is straightforward: The variable ϕ_1 describes a field with mass $\sqrt{\frac{-3\mu^2}{2}} > 0$, the variable ϕ_2 describes a massless field - a so called Goldstone boson. The terms in the fourth line which have not been written explicitly, are cubic and quartic in the fields. They describe self-interactions and interactions between the ϕ_1 and the ϕ_2 field.

4.1.2 The General Case

Let us generalize the considerations from the above subsection to a classical field theory with an n-component real field $\tilde{\phi}(x) = (\tilde{\phi}_1, \cdots, \tilde{\phi}_n)$ described by a Lagrangian of the usual form

$$\tilde{\mathcal{L}}(\partial_\mu\tilde{\phi},\tilde{\phi}) = \sum_{i=1}^{n} \partial_\mu\tilde{\phi}_i(x)\partial^\mu\tilde{\phi}_i(x) - \tilde{V}(\tilde{\phi}(x)). \tag{69}$$

We further assume this Lagrangian to be invariant under the representation of a symmetry group G. In the spirit of the above discussion, we require that the potential be minimized not at $\tilde{\phi}(x) = 0$, but that there be a manifold of minima

$$M = \{\vec{v} : V(\tilde{\phi}(x) = \vec{v}) = \text{min.}\}. \tag{70}$$

Analogously to the discussion above we then choose one particular $\vec{v_0} \in M$ to be the ground state and assume the physical fields to be perturbations around $\vec{v_0}$ instead of around zero.
It is important to note that the manifold M is invariant under the full group G, i.e. if $\vec{v}$ minimizes the potential, then $g\vec{v}$ with $g \in G$ also does. The breaking of the symmetry occurs when we choose one particular vector $\vec{v_0}$ to be the groundstate. This vector $\vec{v_o}$ is usually not invariant under the full group G anymore (hence the notion symmetry breaking) but only under a subgroup $H \subset G$.
This symmetry property can be expressed in terms of the generators of G and H. Let $\{\lambda_1, \cdots, \lambda_k\}$ be the generators of G arranged in such a way that $\{\lambda_1, \cdots, \lambda_l\}$ are the generators of H and $\{\lambda_{l+1}, \cdots, \lambda_k\}$ is the completion to a full basis of Lie(G). We then have

$$\begin{array}{rcll} \lambda_a\vec{v_0} & = & 0 & a = 1, \cdots, l \\ \lambda_a\vec{v_0} & \neq & 0 & a = l+1, \cdots, k \end{array} \tag{71}$$

Let us now return to the Lagrangian of the theory. Introducing the physical fields ϕ by

$$\tilde{\phi}(x) = \vec{v_0} + \phi(x) \tag{72}$$

we may rewrite the Lagrangian

$$\begin{aligned} \mathcal{L}(\partial_\mu\phi,\phi) &= \tilde{\mathcal{L}}(\partial_\mu\phi,\phi+\vec{v_0}) \\ &= \tfrac{1}{2}\sum_{i=1}^{n} \partial_\mu\phi_i(x)\partial^\mu\phi_i(x) - \underbrace{\frac{\partial\tilde{V}}{\partial\tilde{\phi}_i}|_{(\tilde{\phi}=\vec{v_o})}}_{=0}\phi_i - \tfrac{1}{2}\tfrac{\partial^2\tilde{V}}{\partial\tilde{\phi}_i\partial\tilde{\phi}_j}|_{(\tilde{\phi}=\vec{v_o})}\phi_i\phi_j + O(\phi^3) \\ &\to \tfrac{1}{2}\sum_{i=1}^{n} \partial_\mu\phi_i(x)\partial^\mu\phi_i(x) - \tfrac{1}{2}M_{ij}\phi_i(x)\phi_j(x) + O(\phi^3) \end{aligned} \tag{73}$$

As $M_{ij} = \left(\frac{\partial^2 V}{\partial\tilde{\phi}_i\partial\tilde{\phi}_j}\right)$ is symmetric we may assume without loss of generality that it is already diagonal. Furthermore it must be positive semidefinite since we have expanded the potential term around its minimum. Putting things together we find

$$\mathcal{L}(\partial_\mu\phi,\phi) = \sum_{i=1}^{n}\left(\frac{1}{2}\partial_\mu\phi_i(x)\partial^\mu\phi_i(x) - \frac{1}{2}m_i^2\phi_i(x)^2\right) + O(\phi^3) \tag{74}$$

from which it is apparent that the diagonal elements are in fact the masses squared of the physical ϕ- fields.
We can now use the symmetries of the original Lagrangian and of the physical groundstate $\vec{v_0}$ to gain further information about the masses. Recall from above the definition

$$H = \{h \in G : h\vec{v_0} = \vec{v_0}\}. \tag{75}$$

We thus get

$$\mathcal{L}(\partial_\mu h\phi, h\phi) = \tilde{\mathcal{L}}(\partial_\mu h\phi, h\phi + \vec{v_0}) = \tilde{\mathcal{L}}(\partial_\mu h\phi, h(\phi + \vec{v_0})) = \mathcal{L}(\partial_\mu \phi, \phi) \tag{76}$$

i.e. the Lagrangian describing the physical fields - and in particular its potential term $V(\phi = \tilde{V}(\phi + \vec{v_0}))$ - is invariant under the action of the group H. Recalling the generators $\{\lambda_a \quad a = 1, \cdots, l\}$ of the broken symmetry group H we have

$$0 = V((1 - i\theta\lambda_a)\phi) - V(\phi) = -i\theta \left(\frac{\partial V}{\partial \phi}\right) \lambda_a \phi - \frac{\theta^2}{2} \underline{\phi^T \lambda_a^T \left(\frac{\partial^2 V}{\partial \phi^2}\right) \lambda_a \phi} + O(\theta^3). \tag{77}$$

In particular the underlined expression has to vanish for all possible field configurations what can be equivalently stated as

$$\lambda_a^T \left(\frac{\partial^2 V}{\partial \phi^2}\right) \lambda_a = \lambda_a^T M \lambda_a = 0 \qquad a = 1, \cdots, l. \tag{78}$$

From this we can conclude that

$$\dim \ker(M) = n - \dim(\ker(\lambda_1) \cap \cdots \cap \ker(\lambda_l)) = \dim(G) - \dim(H). \tag{79}$$

But dim ker(M) is obviously the number of particles with zero mass in the theory.
We summarize the results we have obtained in a theorem that was originally stated by Goldstone: *In a theory that is spontaneously broken, we have as many massless particles in the spectrum as there are broken generators.*

4.2 The Quantum Mechanical Version of Goldstone's Theorem

In the above section, we tried to understand the ideas behind Goldstone's theorem. However our approach to the problem was purely classical. We will therefore have to reformulate the ideas of the above section in quantum mechanical language.
In analogy to the classical case, for spontaneous symmetry breaking to occur we require

$$\langle 0| \phi(x) |0\rangle \neq 0. \tag{80}$$

Let λ_a be a generator of the symmetry group G of the Lagrangian such that $\langle 0| \lambda_a \phi |0\rangle \neq 0$. From (19) and (21) we have with $|\alpha\rangle = |0\rangle$:

$$\begin{aligned} \langle 0|\,\phi\,|0\rangle &= \langle 0|\,e^{i\theta Q_a}(e^{-i\theta\lambda_a}\phi)e^{-i\theta Q_a}\,|0\rangle \\ &= \langle 0|\,\phi\,|0\rangle + i\theta(\langle 0|\,Q_a\phi - \phi Q_a\,|0\rangle - \underbrace{\lambda_a\,\langle 0|\,\phi\,|0\rangle)}_{\neq 0} \end{aligned} \tag{81}$$

We therefore must necessarily have

$$\hat{Q}^a(t)\,|0\rangle \neq 0. \tag{82}$$

It is worth stopping at this point for a second and carefully recalling what the vacuum of a quantum field theory is: It is a vector in the spectrum of the Hamilton operator that is Lorentz invariant. However $|0\rangle$ need not necessarily be invariant under a symmetry group G of the fields, and a theory is called *spontaneously broken* if this is indeed the case.
(82) is a remarkable result that gives rise to a new interpretation of spontaneous symmetry breaking: *A theory is spontaneously broken if the vacuum is charged.* The generators λ_a that do not leave $\langle 0|\,\phi(x)\,|0\rangle$ invariant are called spontaneously broken. Using (81) this can be reformulated: *A generator is called spontaneously broken if it is associated to a charge that does not annihilate the vacuum.*
Of course there might still be charges that annihilate the vacuum of the spontaneously broken theory. The generators associated to these charges are called unbroken. They generate the symmetry group H as defined in the previous section.
Having defined what the spontaneous symmetry breaking of a quantum field theory means, we can reformulate Goldstone's theorem: *In a quantum field theory that is explicitly broken, each broken generator generates a massless particle in the spectrum.*
The proof of Goldstone's theorem will consist of two parts:

1. We show that there exist certain Green's functions which have a pole at zero momentum.

2. We show that the existence of these poles implies the existence of massless particles.

In order to address the first point, let us consider the Green function

$$G^\mu_{a,i}(x-y) = \langle 0|\,T[J^\mu_a(x)\phi_i(y)]\,|0\rangle\,. \tag{83}$$

The divergence of this Green function is

$$
\begin{aligned}
& \tfrac{\partial}{\partial x^\mu} G^\mu_{a,i}(x-y) \\
= \; & \langle 0| \tfrac{\partial}{\partial x^\mu}[\theta(x^0-y^0)J^\mu_a(x)\phi_i(y) - \theta(y^0-x^0)\phi_i(y)J^\mu_a(x)]\,|0\rangle \\
= \; & \langle 0|\,\theta(x^0-y^0)[\underbrace{\frac{\partial}{\partial x^\mu}J^\mu_a(x)}_{=0}]\phi_i(y) - \theta(y^0-x^0)\phi_i(y)[\underbrace{\frac{\partial}{\partial x^\mu}J^\mu_a(x)}_{=0}]\,|0\rangle \\
& +\tfrac{\partial\theta(x^0-y^0)}{\partial x^0}\langle 0|\,J^0_a(x)\phi_i(y)\,|0\rangle + \tfrac{\partial\theta(y^0-x^0)}{\partial x^0}\langle 0|\,\phi_i(y)J^0_a(x)\,|0\rangle \\
= \; & \delta(x^0-y^0)\,\langle 0| \quad \underbrace{[J^0_a(x),\phi_i(y)]}_{-\delta^{(3)}(\vec{x}-\vec{y})(\lambda_a)_{ij}\phi_j(x)} \quad |0\rangle \\
= \; & -\delta^{(4)}(x-y)(\lambda_a)_{ij}\,\langle 0|\,\phi_j(x)\,|0\rangle = -\delta^{(4)}(x-y)(\lambda_a)_{ij}\,\langle 0|\,\phi_j(0)\,|0\rangle
\end{aligned}
\tag{84}
$$

where we used $\langle 0|\,\phi(x)\,|0\rangle = \langle 0|\,\phi(0)\,|0\rangle)$ (from translational invariance of the vacuum) in the last step.
Expressing this relation in Fourier space where

$$G^\mu_{a,i}(p) = \int \mathrm{d}^4x G^\mu_{a,i}(x)e^{ipx} \tag{85}$$

we find

$$ip_\mu G^\mu_{a,i}(p) = (\lambda_a)_{ij}\,\langle 0|\,\phi_j(0)\,|0\rangle \tag{86}$$

what can be easily solved to give

$$G^\mu_{a,i}(p) = -i\frac{p^\mu}{p^2}(\lambda_a)_{ij}\,\langle 0|\,\phi_j(0)\,|0\rangle \tag{87}$$

Thus we have shown that indeed there exist l (= number of broken generators) Green functions that have simple poles at $p^2 = 0$ and Part 1 is proven.

Let us proceed to Part 2 of the proof. Recall that a free scalar field can always be written in terms of creation and annihilation operators

$$\phi_i(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{\mathrm{d}^3p}{2\omega_i(\vec{p})}\{a^*_i(\vec{p})e^{i(\omega_i(\vec{p})t-\vec{p}\vec{x})} + a_i(\vec{p})e^{-i(\omega_i(\vec{p})t-\vec{p}\vec{x})}\} \tag{88}$$

where $\omega_i(\vec{p}) = \sqrt{\vec{p}^2 + m_i^2}$ depends on the mass of the field ϕ_i. From this formula we get

$$a^*_i(\vec{p}) = -\frac{i}{(2\pi)^{\frac{3}{2}}}\int e^{-i(\omega_i(\vec{p})t-\vec{p}\vec{x})}(i\omega_i(\vec{p}) + \frac{\partial}{\partial t})\phi_i(t,\vec{x})\mathrm{d}^3x \tag{89}$$

We can then compute the matrix element

$$
\begin{aligned}
& \langle 0| J_a^\mu(y) a_i^*(\vec{p}) |0\rangle \\
= \; & -\frac{i}{(2\pi)^{\frac{3}{2}}} \int_{t=-\infty} e^{-i(\omega_i(\vec{p})t-\vec{p}\vec{x})}(i\omega_i(\vec{p}) + \tfrac{\partial}{\partial t}) \langle 0| J_a^0(y)\phi_i(t,\vec{x}) |0\rangle \, \mathrm{d}^3x \\
= \; & \underbrace{\langle 0| a_i^*(\vec{p}) J_a^\mu(y) |0\rangle}_{=0} \\
& + \frac{i}{(2\pi)^{\frac{3}{2}}} \int \tfrac{\partial}{\partial t}\{e^{-i(\omega_i(\vec{p})t-\vec{p}\vec{x})}(i\omega_i(\vec{p}) + \tfrac{\partial}{\partial t}) \langle 0| T[J_a^0(y)\phi_i(t,\vec{x})] |0\rangle\} \mathrm{d}^3x\mathrm{d}t \\
= \; & \frac{i}{(2\pi)^{\frac{3}{2}}} \int e^{-i(\omega_i(\vec{p})t-\vec{p}\vec{x})}(-\triangle - m_i^2) \langle 0| T[J_a^\mu(y)\phi_i(t,\vec{x})] |0\rangle \, \mathrm{d}^3x\mathrm{d}t \\
= \; & \frac{i}{(2\pi)^{\frac{3}{2}}} \int e^{-i(\omega_i(\vec{p})t-\vec{p}\vec{x})}(\vec{p}^2 - m_i^2) \langle 0| T[J_a^\mu(y)\phi_i(t,\vec{x})] |0\rangle \, \mathrm{d}^3x\mathrm{d}t \\
= \; & \frac{i}{(2\pi)^{\frac{3}{2}}}(p^2 - m_i^2) G_{a,i}^\mu(p) e^{-ipy}
\end{aligned}
\tag{90}
$$

where we used the fundamental theorem of calculus in the second equality and the fact that $\phi_i(x)$ satisfies the Klein-Gordon equation

$$
(\partial_\mu \partial^\mu - m_i^2)\phi_i(x) = (\frac{\partial^2}{\partial t^2} - \triangle - m_i^2)\phi_i(x) = 0 \tag{91}
$$

in the third equality. The last equality follows from comparison with (83). We then have obtained two different equations for the Green function

$$
G_{a,i}^\mu = -i\frac{p^\mu}{p^2}(\lambda_a)_{ij} \langle 0| \phi_j(0) |0\rangle = -i(2\pi)^{\frac{3}{2}} \frac{e^{ipy}}{p^2 - m_i^2} \langle 0| J_a^\mu(y) a_i^*(\vec{p}) |0\rangle \tag{92}
$$

from which directly follows, that the mass of the ϕ_i field must vanish, $m_i = 0$. This completes the proof of Goldstone's theorem.

5 Spontaneous Breaking of Chiral Symmetry

5.1 Spontaneous Symmetry Breaking in QCD

Before we can start to consider the consequences of chiral symmetry breaking, let us shortly review why we expect spontaneous symmetry breaking to occur in QCD at all.
In superconductivity, a small attractive interaction between electrons leads to a condensate of Cooper pairs in the ground state. In QCD, we have strong attractive interactions between quarks and antiquarks. By analogy, we assume that there will be a condensate of quark-antiquark pairs in the ground state of the theory:

$$
\langle 0| \overline{q}q |0\rangle = \langle 0| \overline{q}_L q_R + \overline{q}_R q_L |0\rangle \neq 0 \quad q = u, d, s \tag{93}
$$

We know that $\overline{q}q$ is invariant under vector transformations, but not under axial transformations, so we can at this early point already assume that the vector charges remain to be conserved while the axial charges are broken, giving rise to Goldstone bosons.

5.2 The Pion

We introduce eighteen scalar operators

$$\Phi_{ij} = \overline{q}_j q_i \quad \Pi_{ij} = i\overline{q}_j \gamma^5 q_i \qquad i,j \in \{u,d,s\} \tag{94}$$

Under parity ($q(t,\vec{x}) \to \gamma^0 q(t,-\vec{x})$) we find that

$$\Phi_{ij}(t,\vec{x}) \to \Phi_{ij}(t,-\vec{x}) \qquad \Pi_{ij}(t,\vec{x}) \to -\Pi_{ij}(t,-\vec{x}). \tag{95}$$

Further note that

$$\Phi^\dagger(x) = \Phi(x) \qquad \Pi^\dagger(x) = \Pi(x). \tag{96}$$

It is straightforward to verify that under chiral vector transformations

$$\Phi(x) \to e^{-i\theta_a \lambda_a^F} \Phi(x) e^{i\theta_a \lambda_a^F} \qquad \Pi(x) \to e^{-i\theta_a \lambda_a^F} \Pi(x) e^{i\theta_a \lambda_a^F} \tag{97}$$

whereas under chiral axial transformations

$$\Phi(x) \to e^{-i\theta_a \lambda_a^F} \Phi(x) e^{-i\theta_a \lambda_a^F} \qquad \Phi(x) \to e^{-i\theta_a \lambda_a^F} \Phi(x) e^{-i\theta_a \lambda_a^F}. \tag{98}$$

In the spirit of (93) we assume that

$$\langle 0|\, \Phi \,|0\rangle = v_{(3\times 3)} \neq 0. \tag{99}$$

where v is a (3×3)-matrix. Using equation (95) we further have to require

$$\langle 0|\, \Pi \,|0\rangle = 0 \tag{100}$$

if we want the theory to preserve parity.
From the discussion in section 3.4 about chiral symmetry breaking by the quark masses, we know that the chiral vector symmetry is realized approximately in nature. We therefore want it to be preserved in the process of spontaneous symmetry breaking, i.e. we want the full chiral group $SU(3)_{Vector} \times SU(3)_{Axial}$ to be spontaneously broken to $SU(3)_{Vector}$. This means we have to choose the vacuum vector $\vec{v}$ from (99) such that is invariant under $SU(3)_{Vector}$. There remains only one possibility for spontaneous symmetry breaking:

$$\langle 0|\, \Phi \,|0\rangle = v_{(3\times 3)} = v 1_{3\times 3} \qquad\qquad \langle 0|\, \Pi \,|0\rangle = 0. \tag{101}$$

Now define:

$$\phi_a(x) = \frac{1}{2}\mathrm{Tr}(\Phi(x)\lambda_a) = \frac{1}{2}\overline{q}(x)\lambda_a q(x) \quad \pi_a(x) = \frac{1}{2}\mathrm{Tr}(\Pi(x)\lambda_a) = \frac{1}{2}\overline{q}(x)\lambda_a \gamma^5 q(x). \tag{102}$$

The quark wavefunctions satisfy canonical equal-time anti-commutation relations

$$\{q_{i,r}(t,\vec{x}), q_{j,s}(t,\vec{y})\} = \{q^\dagger_{i,r}(t,\vec{x}), q^\dagger_{j,s}(t,\vec{y})\} = 0 \qquad i,j \in \{u,d,s\}$$
$$\{q_{i,r}(t,\vec{x}), q^\dagger_{j,s}(t,\vec{y})\} = \delta_{ij}\delta_{r,s}\delta^{(3)}(\vec{x}-\vec{y}) \qquad r,s \in \{1,2,3,4\} \tag{103}$$

where i,j are the colour indices, r,s the spinor indices. In analogy to equation (17) let us compute

$$[A_a(t), \pi_b(x)] = \frac{-i}{4}\overline{q}(x)\{\lambda_a, \lambda_b\}q(x) = \frac{-i}{4}\mathrm{Tr}[\Phi(x)\{\lambda_a, \lambda_b\}]. \tag{104}$$

Consider a Green function

$$G^\mu_{ab} = \langle 0| \, T[A^\mu_a(x)\pi_b(y)] \, |0\rangle \, . \tag{105}$$

Equation (104) and conservation of the axial current make it straightforward to compute

$$\frac{\partial}{\partial x^\mu} G^\mu_{ab} = \delta(x^0 - y^0) \, \langle 0| \, [A^\mu_a(x), \pi_b(y)] \, |0\rangle = -3i\delta^{(4)}(x-y)\delta_{ab}v \tag{106}$$

The details of the calculation are very similar to (84) and the result is, too. Identities of this form are called Ward identities. This last Ward identity gives us an explicit expression for the Green function:

$$G^\mu_{ab}(p) = \frac{3vp^\mu}{p^2}\delta_{ab} \tag{107}$$

What we have proven is the existence of Green functions that have a pole a zero momentum. Thus we are now really in a position to apply Goldstone's theorem. However we would expect 8 Goldstone bosons to appear. This is not what we can see in nature.
In fact we have been superoptimistic to apply Goldstone's theorem to an approximate symmetry. The existence of divergence-less currents, which are results of exact symmetries only, is essential for the proof of Goldstone's theorem. We have considered the divergences after reinstalling the masses in (58). It can be seen, that the error we make is proportional to the masses of the quarks.
Noting that $m_s \approx 10m_{u,d}$, we can hope to improve our results if we consider the chiral $SU(2) \times SU(2)$ symmetry group of the up- and down-quark only. And indeed we are successful. Goldstone's theorem gives three Goldstone bosons resulting from the broken $SU(2)_{Axial}$ group:

$$\Pi = \begin{pmatrix} \overline{u}\gamma^5 u & \overline{d}\gamma^5 u \\ \overline{u}\gamma^5 d & \overline{d}\gamma^5 d \end{pmatrix} \Rightarrow \begin{pmatrix} \pi_1 = \frac{1}{2}\mathrm{Tr}(\Pi\sigma_1) = \frac{1}{2}(\overline{d}\gamma^5 u + \overline{u}\gamma^5 d) \\ \pi_2 = \frac{1}{2}\mathrm{Tr}(\Pi\sigma_2) = \frac{-i}{2}(\overline{d}\gamma^5 u - \overline{u}\gamma^5 d) \\ \pi_3 = \frac{1}{2}\mathrm{Tr}(\Pi\sigma_3) = \frac{1}{2}(\overline{u}\gamma^5 u - \overline{d}\gamma^5 d) \end{pmatrix} \tag{108}$$

The physical pions are linear combinations of the fields π_1, π_2, π_3:

$$\pi^+ = \overline{d}\gamma^5 u \qquad \pi^- = \overline{u}\gamma^5 d \qquad \pi^0 = \frac{1}{\sqrt{2}}(\overline{u}\gamma^5 u - \overline{d}\gamma^5 d) \tag{109}$$

We have very good agreement with experimental facts:

1. The pions have negative parity as expected from (95).
2. They have small masses of about 140 Mev. In particular they are by far the lightest mesons.
3. They have zero spins, i.e. they are indeed bosons.
4. They form a triplet under $SU(2)_{Isospin}$ as expected from (97).

5.3 Masses for the Pions

Note that Goldstone's theorem - despite its elegance and its intuitive interpretation - poses problems to physicists. Even though we have particles in nature that seem to arise from spontaneous symmetry breaking - such as the pions in QCD or the Z^0,the W^+ and the W^- in electroweak theory - these particles can be very heavy in contradiction to Goldstone's theorem.
Let us sketch how the problem is solved in QCD: What we have ignored in our treatment was the existence of mass terms that *explicitly break* the chiral symmetry. Neither the axial nor the vector current survive this symmetry breaking. Their divergences are listed in (58). The corresponding expressions for the SU(2) instead of the SU(3) symmetry group are:

$$\partial_\mu V_b^\mu = i\overline{q}[M, \frac{\sigma_b^F}{2}]q \qquad \partial_\mu A_b^\mu = i\overline{q}\{M, \sigma_b^F\}\gamma^5 q \tag{110}$$

where

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \tag{111}$$

For simplicity we assume that $m_u = m_d = m$. We then find:

$$\partial_\mu V_b^\mu = im\overline{q}[1, \frac{\sigma_b^F}{2}]q = 0 \qquad \partial_\mu A_b^\mu = im\overline{q}\{1, \frac{\sigma_b^F}{2}\}\gamma^5 q = 2m\pi_b. \tag{112}$$

From the calculations in (84) we see that we have to replace (106) by

$$\frac{\partial}{\partial x^\mu} G_{ab}^\mu = -2i\delta^{(4)}(x-y)\delta_{ab}v + 2m \langle 0| T[\pi_a(x)\pi_b(y)] |0\rangle \tag{113}$$

The new term is the Feynman propagator of the scalar pion fields π_a, a=1,2,3. We now require these fields to be independent and massive, thus this propagator reads:

$$\langle 0| T[\pi_a(x)\pi_b(y)] |0\rangle = C\delta_{ab} \int \frac{i}{p^2 - m_a^2} e^{-ip(x-y)} \mathrm{d}^4 p \tag{114}$$

We have to introduce the parameter C with mass dimension 4, because the π fields are not canonically normalized. Expressing the previous equation in Fourier space we therefore find:

$$p_\mu G_{ab}^\mu(p) = 2v\delta_{ab} - C\frac{2m\delta_{ab}}{p^2 - m_a^2} \tag{115}$$

Choosing the parameters v and C appropriately shifts the pole from $p^2 = 0$ to $p^2 = m_a^2$, the mass of the pions. For this to happen, we need in particular

$$(2v - C\frac{2m}{p^2 - m_a^2})|_{(p^2=0)} = 0 \quad \Rightarrow \quad m_a^2 = C\frac{m}{v} \tag{116}$$

Recall that m_a is the mass of the a-th pion, $m = m_u = m_d$ is the mass of the constituent quarks. Assuming that the pions are equally heavy, we can write:

$$m_{Pion}^2 = C\frac{m_{Quark}}{v} \tag{117}$$

Equation (116) is an interesting result: It shows that the mass squared of the pions is essentially the mass of the constituent quarks divided by the scale that describes the symmetry breaking.

The same procedure fails in the electroweak theory. The particles that arise from spontaneous symmetry breaking are very heavy even though the broken theory is exact. However there is one big difference: The broken symmetry is a gauge symmetry, i.e. it has been promoted to a local symmetry. Goldstone's theorem then has to be replaced by the Higgs mechanism.
We do not have the time to discuss Higgs' theorem in detail here. So let us only consider a very simple example: Let ϕ be a massive, complex field. Let the Lagrangian of the theory be invariant under U(1) rotations of this field. Promoting this symmetry to a local symmetry, we have to introduce a massless, scalar gauge field. Spontaneous breaking of this symmetry then gives

$$\begin{pmatrix} \text{2 massive, scalar fields} \\ \text{1 massless gauge field} \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \text{1 massive, scalar field} \\ \text{1 massive gauge field} \end{pmatrix} \tag{118}$$

5.4 Summary

Let us summarize what we learned about spontaneous symmetry breaking:

1. We defined spontaneous symmetry breaking. It means that the vacuum of a theory is not invariant under the full symmetry group of the Lagrangian, i.e. we have a field operator that has a nonvanishing vacuum-expectation value. We saw a very intuitive consequence in quantum field theories, namely that the vacuum is charged.

2. We introduced and proved Goldstone's theorem. This theorem predicts the emergence of massless Goldstone bosons in theories that are spontaneously broken.

3. We applied the results to chiral symmetry of QCD. We motivated why we expect it to be broken and we identified the emerging Goldstone bosons with the pions.

4. We sketched two methods to make the Goldstone bosons massive. One is to include terms that explicitly break the underlying symmetry, making it an approximate symmetry only. The other one is to promote the symmetry to a gauge symmetry such that Higgs' theorem holds.

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