Euclidean path integral formalism: from quantum mechanics to quantum field theory

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Classical mechanics

- Lagrangian mechanics. Principle of least action:

$$\delta S[L] = 0.$$

- Hamiltonian mechanics. Hamilton's canonical equations

$$\dot{q} = \frac{\partial H(q,p)}{\partial p},$$

$$\dot{p} = -\frac{\partial H(q,p)}{\partial q}.$$

Canonical quantization

- Classical variables, Poisson brackets:

$$\{q_i, p_j\} = \delta_{ij}$$

- Quantum operators, Lie brackets:

$$[X, P] = i\hbar$$

- Away from the principle of least action.
- Feynman: Path integrals founded on the principle of least action, but in this case the action of each possible trajectory plays a crucial role.

Real time

- Pictures:
  - Schrödinger
  - Heisenberg
  - Interaction
- Time evolution, Schrödinger equation

$$i\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t)$$

$$|\psi(t)\rangle = e^{-iH(t-t_0)}\,|\psi(t_0)\rangle$$

- Probability amplitude for a particle to move from y to x within time interval t:

$$\langle x|\, e^{-iHt}\,|y\rangle$$

- Free particle, $H \equiv H_0 = \frac{\vec{p}^2}{2m}$

$$\langle x| e^{-iH_0 t} |y\rangle = \left(\frac{m}{2\pi i t}\right)^{\frac{1}{2}} \exp\left(i\frac{m}{2t}(x-y)^2\right)$$

- Particle in a potential, $H = H_0 + V(x)$, define:

$$U_\epsilon \equiv \exp\left(-iH\epsilon\right) \cong W_\epsilon,$$

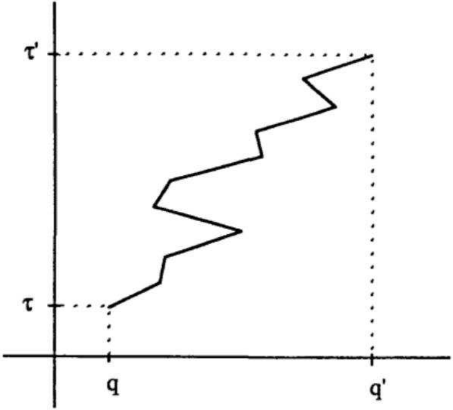
$$\langle x| W_\epsilon |y\rangle = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{1}{2}} \exp\left(i\frac{m}{2\epsilon}(x-y)^2 - i\frac{\epsilon}{2}\left(V(x)+V(y)\right)\right)$$

- W_ϵ as the approximated time evolution operator if $\epsilon = \frac{t}{N}$ is small, because

$$\exp\left(-i\left(H_0+V\right)t\right) = \lim_{N\to\infty} {W_\epsilon}^N.$$

- Inserting $N-1$ complete sets of position eigenstates:

$$\left\langle x \left| e^{-iHt} \right| y \right\rangle = \lim_{N\to\infty} \int dx_1 \cdots dx_{N-1} \left\langle x \right| W_\epsilon \left| x_1 \right\rangle \cdots \left\langle x_{N-1} \right| W_\epsilon \left| y \right\rangle .$$



- Rewrite as

$$\left\langle x \left| e^{-iHt} \right| y \right\rangle = \int Dx e^{iS_\epsilon},$$

- where

$$Dx = \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} dx_1 \cdots dx_{N-1}$$

- and

$$S_\epsilon = \frac{m}{2\epsilon}\left((x-x_1)^2 + \ldots + (x_{N-1}-y)^2\right)$$
$$-\epsilon\left(\frac{1}{2}V(x) + V(x_1) + \ldots + V(x_{N-1}) + \frac{1}{2}V(y)\right)$$

- $S_\epsilon \longrightarrow S$, the action:

$$S = \int_0^t dt' \left(\frac{m}{2} \dot{x}^2 - V(x) \right)$$

- Quantum mechanics amplitude as integral over all paths weighted by e^{iS}.
- Quantum mechanical operators $\Rightarrow$ infinite-dimensional integral.
- Green functions:

$$G(q', t'; q, t) = \int dq'' G(q', t'; q'', t'') G(q'', t''; q, t)$$

Euclidean time

- Goal: Substitute e^{iS} with a real, non oscillating and non-negative function.
- Define imaginary (or Euclidean) time τ as

$$t = -i\tau, \ \tau > 0.$$

- Probability amplitude:

$$\langle x| \, e^{-H\tau} \, |y\rangle = \int Dx e^{-S_E},$$

- where

$$S_E = \int_0^t d\tau' \left(\frac{m}{2}\dot{x}^2 + V(x)\right)$$

is the Euclidean action.

- Every path is weighted by e^{-S_E}.
- Action

$$S = \int_0^t dt' \left(\frac{m}{2}\dot{x}^2 - V(x) \right)$$

and Euclidean action

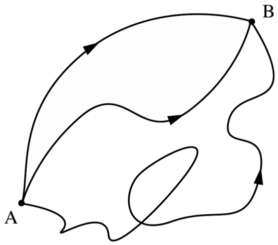
$$S_E = \int_0^t d\tau' \left(\frac{m}{2}\dot{x}^2 + V(x) \right)$$

are related by

$$S\big|_{t=-i\tau} = iS_E,$$

substitution $\frac{d}{dt} = i\frac{d}{d\tau}$ and $dt' = -id\tau'$.

- Classical mechanics: The path is given by $\delta S_E = 0$
- Quantum mechanics: All paths are possible.



- The paths which minimize S_E, this means near $\delta S_E = 0$, are the most likely.
- Physical units: $\exp\left(-S_E/\hbar\right)$
- Classical limit: $\hbar \to 0 \Rightarrow$ least action.

Vacuum's expectation values

- Goal: Elegant expression for $\langle 0| A |0\rangle$.
- Let $|n\rangle$ be the energy eigenstates, then
  - $$Tr\left(e^{-H\tau}A\right) = \sum_{n=0}^{\infty} e^{-E_n\tau} \langle n| A |n\rangle$$
  - and
    $$Z(\tau) = Tr\left(e^{-H\tau}\right) = \sum_{n=0}^{\infty} e^{-E_n\tau}$$
- For $\tau \to \infty$ the term E_0 dominates the sums, this means
  $$\langle 0| A |0\rangle = \lim_{\tau\to\infty} \frac{Tr\left(e^{-H\tau}A\right)}{Z(\tau)}$$
- Mean in a canonical statistical ensemble.

Example: Correlation functions

- $$\langle x(t_1)\cdots x(t_n)\rangle \equiv \langle 0|\, x(t_1)\cdots x(t_n)\,|0\rangle$$
- Continued analytically to Euclidean times $\tau_k = it_k$
- $$\langle x(t_1)\cdots x(t_n)\rangle = \lim_{\tau\to\infty}\frac{1}{Z(\tau)}\int Dx\; x(\tau_1)\cdots x(\tau_n)\exp\left(-S_E\left[x(\tau)\right]\right)$$

  where
  $$Z(\tau) = \int Dx\;\exp\left(-S_E\left[x(\tau)\right]\right).$$

Euclidean space-time

- Goal: Show that, using the imaginary time, we move from a Minkowski to a Euclidean space.
- We define the time coordinate x^4 as

$$x^0 = -ix^4, \;\; x^4 \in \mathbb{R}.$$

- $\left(x^0, x^1, x^2, x^3\right) \rightarrow$ Minkowski (signature $(-1, 1, 1, 1)$)
  $\left(x^1, x^2, x^3, x^4\right) \rightarrow$ Euclidean, because $g\left(x^4, x^4\right) = 1$.

Why quantum field theory?

- Quantum mechanics and special relativity $\Rightarrow$ problems
  - Negative energy states ($E^2 = p^2 + m^2$)
  - Assume existence of antiparticles.
- Quantum field theory:
  - Wave functions replace by field operators.
  - Field operators can create and destroy an infinite number of particles
    $\Rightarrow$ QFT can deal with many particle system.

Wightman and Schwinger functions

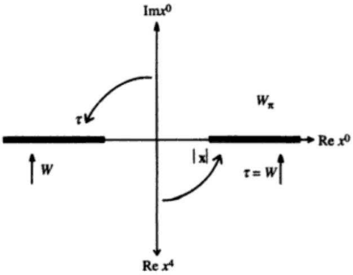
- Define the Wightman function as the n-point correlation functions:

$$W(x_1, \ldots, x_n) = \langle 0| \phi(x_1) \cdots \phi(x_n) |0\rangle$$

- Wightman functions can be continued analytically into a region of the complex plane
- Define the Schwinger functions as

$$S(\ldots; \vec{x}_k, x_k^4; \ldots) \equiv W(\ldots; -ix_k^4, \vec{x}_k; \ldots),$$

for $x_1^4 > x_2^4 > \ldots > x_n^4$.



- Schwinger functions in Euclidean space
  $\Rightarrow$ Wightman functions in Minkowski space.

$$W(x_1, \ldots, x_n) = \lim_{\epsilon_k \to 0, \epsilon_k - \epsilon_{k+1} > 0} S\left(\ldots; \vec{x}_k, ix_k^0 + \epsilon_k; \ldots\right)$$

- Advantages:
  - Schwinger functions obey to simpler properties.
  - From symmetry property we build the representation in terms of functional integrals.
    (Path integral formalism in quantum field theory)

Time-ordered Green functions

- We define the Time-ordered Green functions as

$$\tau\left(x_1,\ldots,x_n\right)=\left\langle 0\right|T\phi\left(x_1\right)\cdots\phi\left(x_n\right)\left|0\right\rangle,$$

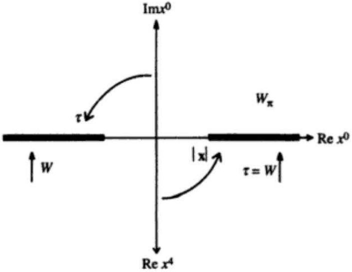
  where T is the time ordering operator.
- For the 2-point function we have

$$\tau\left(x\right)\equiv\tau\left(x,0\right)=\begin{cases}W\left(x\right) & ,x^0>0\\ W\left(-x\right) & ,x^0<0.\end{cases}$$

  - $W\left(x\right)$ is analytic in the lower half complex plane.
  - $W\left(-x\right)$ is analytic in the upper half complex plane.

Wick rotation

- Counter-clockwise rotation



- Generalization:

$$\tau\left(x_1,\ldots,x_n\right)=\lim_{\phi\longrightarrow\frac{\pi}{2}} S\left(\ldots;\vec{x}_k,e^{i\phi}x_k^0;\ldots\right)$$

Path integral formalism in quantum field theory

- Symmetry of Euclidean correlation functions
  $\Rightarrow$ Euclidean fields commute
  - Like classical fields.
  - Random variables, not operators
- Expectation values:

$$\langle F[\phi]\rangle = \int d\mu F[\phi]$$

- For Euclidean functional integral:

$$d\mu = \frac{1}{Z} e^{-S[\phi]} \prod_x d\phi(x)$$

- Combining, like the Euclidean path integral formalism.

Bosonic field theory

- Goal: Find explicitly the expression for the Euclidean correlation $\langle\phi(x_1)\cdots\phi(x_n)\rangle$ in Euclidean functional integral formalism.
- From the Gaussian integrals we find

$$Z_0(J) \equiv \frac{1}{Z_0}\int d^k\phi \exp\left(-\frac{1}{2}(\phi, A\phi) + (J,\phi)\right) = \exp\left(\frac{1}{2}(J, A^{-1}J)\right)$$

where J is an arbitrary vector and

$$Z_0 \equiv \int d^k\phi \exp\left(-\frac{1}{2}(\phi, A\phi)\right).$$

- For a free field we have

$$Z\left[J\right] \equiv Z_0\left[J\right] = \exp\left(\frac{1}{2}\int d^4x d^4y J\left(x\right) G\left(x,y\right) J\left(y\right)\right),$$

where $G(x,y)$ is the propagator that satisfies

$$\left(\Box + m^2\right) G\left(x,y\right) = \delta\left(x-y\right).$$

- The correlation functions are given by

$$\left\langle \phi\left(x_1\right)\cdots\phi\left(x_{2n}\right)\right\rangle = \left.\frac{\delta^n Z\left[J\right]}{\delta J\left(x_1\right)\cdots\delta J\left(x_n\right)}\right|_{J=0}.$$

- We move from a discrete to a continuous representation, this means
  - $\phi_i \to \phi(x)$,
  - $\frac{\partial}{\partial J_i} \to \frac{\delta}{\delta J(x)}$.
- Comparing

$$Z_0[J] = \exp\left(\frac{1}{2}(J, GJ)\right),$$

$$Z_0[J] = \exp\left(\frac{1}{2}(J, A^{-1}J)\right),$$

we find $A \to G^{-1} = \Box + m^2$.

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$$\frac{1}{2}(\phi, A\phi) \to \frac{1}{2}(\phi, G^{-1}\phi) = \frac{1}{2}(\phi, (\Box + m^2)\phi) = S_0[\phi].$$

- Inserting in the Gaussian integrals we find

$$Z_0\left[J\right] = \frac{1}{Z_0}\int\prod_x d\phi\left(x\right)\exp\left(-S_0+\left(J,\phi\right)\right),$$

where

$$Z_0 = \int\prod_x d\phi\left(x\right)\exp\left(-\frac{1}{2}\left(\phi,\left(\Box+m^2\right)\phi\right)\right).$$

- Infinite-dimensional integral $\Rightarrow$ not well defined.
- Define the measure

$$d\mu_0\left(\phi\right) = \frac{1}{Z_0}D\left[\phi\right]e^{-S_0(\phi)},$$

where

$$D\left[\phi\right] \equiv \prod_x d\phi\left(x\right).$$

- Using this functional approach we find the correlation functions

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int d\mu_0(\phi)\, \phi(x_1) \cdots \phi(x_n) =$$

$$\frac{1}{Z_0} \int \prod_x d\phi(x)\, e^{-S_0[\phi]} \phi(x_1) \cdots \phi(x_n).$$

Interacting field

- We neglect the problems associated with divergences and renormalization.
- Euclidean action

$$S[\phi] = S_0[\phi] + S_I[\phi],$$

where S_I describes the interaction part.
- From Gaussian functional integral we find

$$Z[J] = \frac{1}{Z} \int \prod_x d\phi_{in}(x)\, e^{-S[\phi]+(J,\phi)}$$

where

$$Z = \int \prod_x d\phi_{in}(x)\, e^{-S[\phi]},$$

and ϕ_{in} is the free field.

Connection with perturbative expansion

- Goal: Deduce the rules of perturbation theory from functional integral.
  (Feynman rules)
- Consider a scalar field with a quartic self-interaction given by

$$S_I[\phi] = \frac{g}{4!}\int d^4x\ \phi(x)^4.$$

- We start to expand e^{-S_I} in the functional expression for an interacting field.

- We find the functional integral

$$Z[J] = \frac{Z_0}{Z} \exp\left(S_I\left[\frac{\delta}{\delta J(x)}\right]\right) \exp\left(\frac{1}{2}(J, G_0 J)\right).$$

- For the associated correlations functions we find

$$G(x_1, \ldots, x_n) = \frac{Z_0}{Z} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \exp\left(S_I\left[\frac{\delta}{\delta J(x)}\right]\right) \cdot \exp\left(\frac{1}{2}(J, G_0 J)\right)\Bigg|_{J=0}.$$

Graphical interpretation of the last equation:

- Expand the exponential of $S_I\left[\frac{\delta}{\delta J(x)}\right]$
  - Each derivative $\frac{\delta}{\delta J(x_i)}$ is indicated by an external point from which emerges a line

  - Each factor $-g\int d^4x\left(\frac{\delta}{\delta J(x_i)}\right)^4$ is indicated by an internal vertex, from which emerges four lines.

- Expand $\exp\left(\frac{1}{2}\left(J, G_0 J\right)\right)$
  - Internal lines

- We can normalize the generating functional to $Z(0) = 1$, using

$$\frac{Z}{Z_0} = \exp\left(-S_I\left[\frac{\delta}{\delta J(x)}\right]\right)\exp\left(\frac{1}{2}(J, G_0 J)\right)\Bigg|_{J=0}.$$

- Graphical interpretation:
  - No external points.
  - Vacuum graphs.
  - These terms cancel out in the graphical representation of Green's functions.
- Cancel out only vacuum graphs, but not all internal loops (divergences)

- This divergences imply that the functional $Z[J]$ is not well-defined.
- Regularization allows us to evaluate this functional integral, separating the divergent part.
- Renormalization absorbs the divergences by a redefinition of the parameter of the theory.
- The renormalized values of the parameters are considered as the physical ones.
- Renormalizable theory: at every order in perturbation theory, only a finite number of parameters needs to be renormalized.