Proseminar FS09 in Theoretical Physics - Perturbative and non-perturbative methods for strong interactions

Euclidean path integral formalism: from quantum mechanics to quantum field theory

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1 Introduction

In this first topic I will introduce the Euclidean path integral formalism, because this is a fundamental instrument for a non-perturbative approach to quantum field theory. We will start with the Feynman path integrals for a single particle in quantum mechanics and then we will extend to imaginary time. Later we will deal with the Euclidean path integrals for a real scalar field, this means for a bosonic field, and we will show the analogy between this formalism and statistical mechanics. At the end we will see the connection with the perturbative theory.

We first introduce the Euclidean path integrals in quantum mechanics for completeness, but this formalism becomes relevant for quantum field theory. In fact in quantum mechanics we have already other methods that are more efficient and convenient to solve the typical problems, for example scattering amplitudes, bound state energies or eigenfunctions of the hydrogen atom. Moreover the only non-trivial example, that we can solve analytically with the Feynman path integrals, is the harmonic oscillator. For this reason we can see the Euclidean path integral formalism simply like another elegant way to describe the time evolution of a quantum system. Instead in quantum field theory it has become the fundamental instrument to study particle physics. It has been essential to build the gauge theories, like quantum electrodynamics or quantum chromodynamics.

Now, before beginning with the path integral formalism, I would remember some fundamental ideas of classical mechanics and in this way I think that it is possible to understand the Feynman's motivations to introduce a new quantum formalism in which the action of each possible trajectory plays a crucial role. We know that there are, besides the Newton framework, two equivalent formalisms in classical mechanics: the Lagrangian and the Hamiltonian formalisms. In the first we define the Lagrangian as

$$L(q_i, \dot{q}_i, t) = T - V, \tag{1}$$

where T is the kinetic term and V the potential term. At this point the equations of motion are given by the principle of least action[1]

$$\delta S[L] = 0. \tag{2}$$

Instead for the Hamiltonian formalism, dynamics is founded on the Hamil-

[1] The action is defined by $S[L] = \int L dt$ and we know that the variational principle is equivalent, for variations that vanish at the initial and end points, to Euler-Lagrange equations.

tonian[2] and in particular on the Hamilton's canonical equations

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \quad (3)$$

$$\dot{p} = -\frac{\partial H(q,p)}{\partial q}. \quad (4)$$

The Hamiltonian formalism is the natural framework to move from the classical mechanics to quantum mechanics. Indeed in the canonical quantization of a system we take the Hamiltonian and we have to substitute the classical variables, which satisfy the commutation's rule with the Poisson brackets[3], with the quantum operators, which have to satisfy the commutation's rule with the Lie brackets. In this way we move away from the principle of least action, for this reason, introducing the Feynman path integral formalism, we can again found all dynamics of the system on this fundamental principle that leads physics and hence, the Lagrangian being the fundamental quantity, this approach preserves the symmetries of a theory. We will see that in quantum mechanics the possible paths are not only those such that $\delta S[L] = 0$, but we can say that in a classical limit these paths are simply the most probable.

2 Path integral formalisms in quantum mechanics

We will start with the Feynman path integrals, this means that we will use the real time, and then, in a second step, we will replace it with the imaginary time. We will show that this method will allow us to move from a space-time described by the Minkowski metric to a Euclidean space-time. Moreover we will see that the Euclidean formalism has some other convenient advantages.

2.1 Real time

In quantum mechanics we can use more equivalent pictures to describe the time evolution. In Schrödinger picture we have that the operators are time independent and all the time dependence is stored in the state of the system.

[2]In a rigorous way the Hamiltonian $H(q,p)$ is defined by the Legendre transformation of the Lagrangian $L(q,\dot{q},t)$ with respect to canonical conjugate momentum, defined as $p = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}$.

[3]This means that generalized coordinates and the canonical conjugate momenta satisfy $\{q_i, p_j\} = \delta_{ij}$.

Instead in the Heisenberg picture we put the time evolution in the operators and we have the state that is time independent. Another possibility is to use the interaction picture, where we put the trivial time evolution in the operators and the state evolution depends on the interaction.

Now we use the Schrödinger picture such that the time evolution of the state[4] is given by the Schrödinger equation

$$i\frac{\partial \psi(x,t)}{\partial t} = H\psi(x,t), \tag{5}$$

and we find[5]

$$|\psi(t)\rangle = e^{-iH(t-t_0)} |\psi(t_0)\rangle . \tag{6}$$

Now, using the eigenstate $|y\rangle$ of the position operator, we can find that the probability amplitude for a particle to move from y to x within time interval t is given by

$$\langle x| e^{-iHt} |y\rangle . \tag{7}$$

For a free particle we have

$$H \equiv H_0 = \frac{\vec{p}^2}{2m} \tag{8}$$

and we can find[6] explicitly the amplitude probability

$$\langle x| e^{-iH_0 t} |y\rangle = \left(\frac{m}{2\pi i t}\right)^{\frac{1}{2}} \exp\left(i\frac{m}{2t}(x-y)^2\right) . \tag{9}$$

Instead for a particle in a potential we have

$$H = H_0 + V(x) \tag{10}$$

and in general we can not write explicitly the probability amplitude, but it is possible to define the time evolution operator for small times ϵ as

$$U_\epsilon \equiv \exp(-iH\epsilon) . \tag{11}$$

Using the Baker-Campbell-Hausdorff formula, this means $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$, we can approximate the time evolution operator U_ϵ with

$$W_\epsilon = \exp\left(-iV\frac{\epsilon}{2}\right)\exp(-iH_0\epsilon)\exp\left(-iV\frac{\epsilon}{2}\right) \tag{12}$$

[4]We remember that the state is described by a vector in the Hilbert space $\mathcal{H}$ and the observables by Hermitian operators acting on the Hilbert space $\mathcal{H}$.

[5]We use $\hbar = 1$ and we restrict to a time independent Hamiltonian.

[6]We have to use $\int dp\,|p\rangle\langle p| = \mathbf{1}$ and $\int \frac{dp}{2\pi} e^{ip(x-y)} \exp\left(-i\frac{p^2}{2m}t\right) = \left(\frac{m}{2\pi i t}\right)^{\frac{1}{2}} \exp\left(i\frac{m}{2t}(x-y)^2\right)$.

and we find its matrix elements, that are given by

$$\langle x|\, W_\epsilon \,|y\rangle = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{1}{2}} \exp\left(i\frac{m}{2\epsilon}\,(x-y)^2 - i\frac{\epsilon}{2}\,(V(x)+V(y))\right). \qquad (13)$$

We can show[7] that

$$\exp\left(-i\,(H_0+V)\,t\right) = \lim_{N\to\infty} {W_\epsilon}^N \qquad (14)$$

and this means that we can use the approximated time evolution operator W_ϵ if the time interval t is divided into small elements

$$\epsilon = \frac{t}{N}. \qquad (15)$$

At this point, we know an approximated time evolution operator W_ϵ, which becomes exact in the limit $\epsilon \to 0$ or equivalently $N \to \infty$. For this reason, inserting $N-1$ complete sets of position eigenstates we find

$$\left\langle x \left| e^{-iHt} \right| y \right\rangle = \lim_{N\to\infty} \int dx_1 \cdots dx_{N-1}\, \langle x|\, W_\epsilon \,|x_1\rangle \cdots \langle x_{N-1}|\, W_\epsilon \,|y\rangle. \qquad (16)$$

Knowing the matrix elements we can rewrite this expression as

$$\left\langle x \left| e^{-iHt} \right| y \right\rangle = \int Dx\; e^{iS_\epsilon}, \qquad (17)$$

where

$$\begin{aligned} S_\epsilon &= \frac{m}{2\epsilon}\left((x-x_1)^2 + \ldots + (x_{N-1}-y)^2\right) \\ &\quad -\epsilon\left(\frac{1}{2}V(x) + V(x_1) + \ldots + V(x_{N-1}) + \frac{1}{2}V(y)\right) \end{aligned} \qquad (18)$$

and

$$Dx = \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} dx_1 \cdots dx_{N-1}. \qquad (19)$$

This expression S_ϵ is an approximation to the action S of a particle moving from point y to another point x along a path $x(t)$ with $x_k = x(k\epsilon)$

$$S = \int_0^t dt' \left(\frac{m}{2}\dot{x}^2 - V(x)\right). \qquad (20)$$

[7] Starting from ${U_\epsilon}^N - {W_\epsilon}^N = \sum_{k=0}^{N-1} {U_\epsilon}^k\,(U_\epsilon - W_\epsilon)\,{W_\epsilon}^{N-1-k}$ and using the matrix norm $(\|F\cdot G\| \le \|F\|\cdot\|G\|\,, \|F+G\| \le \|F\| + \|G\|)$ we can find $\lim_{N\to\infty}\left\|{U_\epsilon}^N - {W_\epsilon}^N\right\| = 0$.

Equation (17) describes the quantum mechanical probability amplitude and we can interpret[8] it as an integral over all paths weighted by e^{iS}. We see that in this new formalism we have eliminated the quantum mechanical operators in favour of an infinite-dimensional integral. In the next section we will introduce the imaginary time, and then we will see a better interpretation of this integration over all possible paths.

We can find another approach if we interpret this probability amplitude as the Green function that describes the time evolution of the state. Indeed, using $\int dq \left|q\right\rangle \left\langle q\right| = \mathbf{1}$, we can write

$$\psi(q', t') = \int dq G(q', t'; q, t)\psi(q, t), \tag{21}$$

where $G(q', t'; q, t) = \left\langle q'\right| e^{-iH(t'-t)} \left|q\right\rangle$. In this framework the path integral formalism is founded directly on the following convolution rule,

$$G(q', t'; q, t) = \int dq'' G(q', t'; q'', t'')G(q'', t''; q, t). \tag{22}$$

2.2 Euclidean time

Now we introduce the imaginary time and then, we will show that this choice implies that the space-time is described by a Euclidean metric.

We define the imaginary (or Euclidean) time τ as

$$t = -i\tau, \ \ \tau > 0. \tag{23}$$

Using the imaginary time τ the time evolution operator[9] becomes

$$\exp\left(-H\tau\right). \tag{24}$$

If we do the same steps as for real time, this means we have to define an approximated time evolution operator W_ϵ such that we can compute its matrix

[8] The right-hand of the equation (17) can also be written as

$$\sum_{all\ paths} e^{i(phase)},$$

where $\int Dx\left(t\right)$ is another way to write "sum over all paths". Moreover we see from equation (19) that in the limit $N \to \infty$ we have an integration over a continuous space of functions, hence the integrand of right-hand of the equation (17) is a functional, in fact it associates any path $x\left(t\right)$ with a complex phase. For this reason it is also called functional integral formalism. However this is only a theoretical remark, because we need to write the action over a discrete representation of the smooth path $x\left(t\right)$.

[9] We see directly that it is well defined if the potential term V is bounded from below.

elements, and then compute explicitly the probability amplitude, we find

$$\langle x| e^{-H\tau} |y\rangle = \int Dx\ e^{-S_E}, \tag{25}$$

where

$$S_E = \int_0^t d\tau' \left(\frac{m}{2}\dot{x}^2 + V(x)\right) \tag{26}$$

is the Euclidean action[10] and

$$Dx = \lim_{N\to\infty} \left(\frac{m}{2\pi\epsilon}\right)^{\frac{N}{2}} dx_1 \cdots dx_{N-1}. \tag{27}$$

We can show that the classical action and the Euclidean action are related by

$$S|_{t=-i\tau} = iS_E. \tag{28}$$

In fact it is enough to replace $\frac{d}{dt} = i\frac{d}{d\tau}$ and $dt' = -id\tau'$ in the action S.

Now we see that the equation (25) is real and every path is weighted by e^{-S_E}. This approach is advantageous with respect to the complex Feynman path integrals, because we haven't an oscillating function. In this formalism the connection with the classical principle of least action is manifest, but there is a fundamental difference between the classical and the quantum approaches. Indeed we see that in the Euclidean path integrals all paths are possible and those with S_E extremal, this means near $\delta S_E = 0$, are simply more likely. So far we have always used Natural units, in which $\hbar = c = 1$. If we use physical units and reinstate $\hbar$ explicitly, then the weight of every path should read: $\exp(-S_E/\hbar)$ (so that $\hbar$ would play a role similar to the temperature in a Statistical Mechanics system). The classical limit is obtained when $\hbar \to 0$; in fact, in that limit, only the configurations of least action contribute, while all configurations corresponding to larger action values are suppressed.

As we see from equation (27) we have the Euclidean action (26) only in limit $N \to \infty$, but without this limit we have a discretization of the quantum mechanical amplitude on a Euclidean time-lattice with spacing ϵ. On this time-lattice we define the time evolution operator[11] as the transfer matrix[12]

$$\mathbf{T} = \exp\left(-V\frac{\epsilon}{2}\right)\exp\left(-H_0\epsilon\right)\exp\left(-V\frac{\epsilon}{2}\right). \tag{29}$$

[10] Notice the positive sign, that is consistent with equation (28).

[11] Analogous to (12).

[12] The matrix elements are given by
$\langle x| T |y\rangle = \left(\frac{m}{2\pi\epsilon}\right)^{\frac{1}{2}} \exp\left(-\frac{m}{2\epsilon}(x-y)^2 - \frac{\epsilon}{2}(V(x)+V(y))\right).$

We can define the Hamiltonian H_ϵ corresponding to transfer matrix T as

$$\mathbf{T} = \exp\left(-\epsilon H_\epsilon\right), \tag{30}$$

and H_ϵ becomes the Hamiltonian H in the limit $\epsilon \to 0$.

Now, we search a convenient way to express the expectation values $\langle 0| A |0\rangle$ of an operator A in the energy ground state[13] as Euclidean path integrals. This approach becomes important also for the next section, when we deal with the bosonic field and the expectation values on the vacuum. We start from

$$Tr\left(e^{-H\tau}A\right) = \sum_{n=0}^{\infty} e^{-E_n\tau} \langle n| A |n\rangle \tag{31}$$

and

$$Z(\tau) = Tr\left(e^{-H\tau}\right) = \sum_{n=0}^{\infty} e^{-E_n\tau}. \tag{32}$$

We see that for $\tau \to \infty$ the ground state term E_0 dominates the sums and therefore it follows directly that

$$\langle 0| A |0\rangle = \lim_{\tau\to\infty} \frac{Tr\left(e^{-H\tau}A\right)}{Z(\tau)}. \tag{33}$$

We notice that the equation (33) has the same form of the mean in a canonical statistical ensemble. This will be important in quantum field theory, because it will allow us to use the same methods of statistical physics. A special case of equation (33) are the correlation functions[14]

$$\langle x(t_1)\cdots x(t_n)\rangle \equiv \langle 0| x(t_1)\cdots x(t_n) |0\rangle \tag{34}$$

continued analytically to Euclidean times $\tau_k = it_k$, where we find[15]

$$\langle x(t_1)\cdots x(t_n)\rangle = \lim_{\tau\to\infty} \frac{1}{Z(\tau)} \int Dx\; x(\tau_1)\cdots x(\tau_n) \exp\left(-S_E\left[x(\tau)\right]\right), \tag{35}$$

with

$$Z(\tau) = \int Dx\;\; \exp\left(-S_E\left[x(\tau)\right]\right). \tag{36}$$

[13]Let $|i\rangle$ be the eigenstate with eigenvalue E_i, in ascending order, of the Hamiltonian H_ϵ. This means that E_0 is the energy ground state.

[14]We can see these correlation functions as the probability for a particle to pass through $x(t_1)\dots x(t_n)$.

[15]We use $S_E\left[x(\tau)\right]$ to denote a functional.

To end this section we show that using the imaginary time it follows directly that the metric of space-time becomes Euclidean. Indeed we set the coordinate time

$$x^0 = -ix^4,\ x^4 \in \mathbb{R}. \tag{37}$$

Now, if the metric for the coordinates (x^0, x^1, x^2, x^3) was the Minkowski metric, then the metric write with respect to the coordinates (x^1, x^2, x^3, x^4) is a Euclidean metric. In fact, if we use the following signature $(-1, 1, 1, 1)$ for the Minkowski metric, we have $g_{44} = g(x^4, x^4) = g(ix^0, ix^0) = -g_{00} = 1$.

3 Euclidean rotation

For the moment we have seen the Euclidean path integral formalism in quantum mechanics. But we know that we can not bound our analysis to quantum mechanics, because if we want to make it compatible with the special relativity we find some problems. In fact if we use the following relation between energy and momentum, $E^2 = p^2 + m^2$, we find that some particles could have also negative energy states. This problem can be solved assuming the existence of some particles, which move back in the time and which have the opposite quantum numbers. These are named antiparticles and they emerge naturally combining quantum mechanics and special relativity. Quantum mechanics is not a convenient way to work with many particles, but it is better to use quantum field theory, that can describe system with an infinite number of particles. In fact, moving from quantum mechanics to quantum field theory, we replace the wave functions with the field operators, that can create and destroy an infinite number of particles.

Now, before starting with the Euclidean path integral formalism in quantum field theory, we have to study the way to move from a field theory in space-time with the Minkowski metric to a Euclidean space-time, as in the previous section we have continued the real time to imaginary time.

We start to define the Wightman functions as the n-point correlation functions of the scalar field ϕ,

$$W(x_1, \ldots, x_n) = \langle 0| \phi(x_1) \cdots \phi(x_n) |0\rangle . \tag{38}$$

These Wightman functions become important in quantum field theory, because we will find a connection between them and the time-ordered Green functions, that contain all physical information[16]. Now, if we write the field

[16]In the first part of this topic we don't use a formalism founded on the Green functions, but if we see the end of the chapter (2.1), we notice that the information was already stored in the Green functions.

operator as[17]

$$\phi(x) = e^{iP\cdot x}\phi(0)\, e^{-iP\cdot x}, \tag{39}$$

where P is generator of translations and if we continue x_k as

$$x_k = u_k - iy_k, \tag{40}$$

we find, using the spectrum condition[18], that for $y_k - y_{k+1} \in \overline{V}_+$, this means in the forward light cone, the Wightman functions can be continued analytically into this region. Hence we define the Schwinger functions (or Euclidean Green functions) as

$$S\left(\ldots; \vec{x}_k, x_k^4; \ldots\right) \equiv W\left(\ldots; -ix_k^4, \vec{x}_k; \ldots\right), \tag{41}$$

for[19] $x_1^4 > x_2^4 > \ldots > x_n^4$.

Now we show that the Wightman functions can be analytically extended into whole complex x^0-plane, without the real line where $|x^0| > |\vec{x}|$. Hence we can deal with Schwinger functions, that obey simpler properties, and however we can reconstruct the Wightman functions in Minkowski space from the Schwinger functions in the Euclidean space. The symmetry properties of these functions will allow us to find a representation in terms of functional integrals[20]. For simplicity we restrict to the 2-point function. From the definition of the Wightman functions and from A2 we can define the 2-point function as

$$W(x_1, x_2) \equiv W(x_1 - x_2). \tag{42}$$

We note that[21] $W(x)$ is analytic in the lower half complex plane and analogously for $W_\pi(x) \equiv W(-x)$ in the upper half plane. At this point it is important to remember that the real space is described by the Minkowski space, this means by the real x^0 axis. Since $W(x_1, x_2) = W(x_2, x_1)$ we have that W and W_π are equal for a space-like coordinate $x = x_1 - x_2$. Hence they are equal on an open region and this implies that they form a single analytic function in the union of their domains of definition. Extending to n-point functions we see that the Schwinger functions are defined on all non-coinciding Euclidean points $x_j \neq x_k$ for $j \neq k$.

[17] We write $\cdot$ for the scalar product with respect to the Minkowski metric.

[18] See A3 in Appendix $(A.1)$.

[19] We can show that the Euclidean points $x_k = \left(-ix_k^4, \vec{x}_k\right)$, with $x_k^4 \in \mathbb{R}$, $\vec{x}_k \in \mathbb{R}^3$ and $x_k^4 - x_{k+1}^4 > 0$ are in the analytical region.

[20] If we see the footnote (8), we understand that this representation corresponds to the path integral formulation.

[21] See definition (37).

In the 2-point function example we have seen that we get the Wightman functions from the Schwinger functions if they approach the x^0 real axis. Hence, if we generalize this behaviour to the n-point functions, we find

$$W\left(x_1, \ldots, x_n\right) = \lim_{\epsilon_k \to 0, \epsilon_k - \epsilon_{k+1} > 0} S\left(\ldots; \vec{x}_k, ix_k^0 + \epsilon_k; \ldots\right), \tag{43}$$

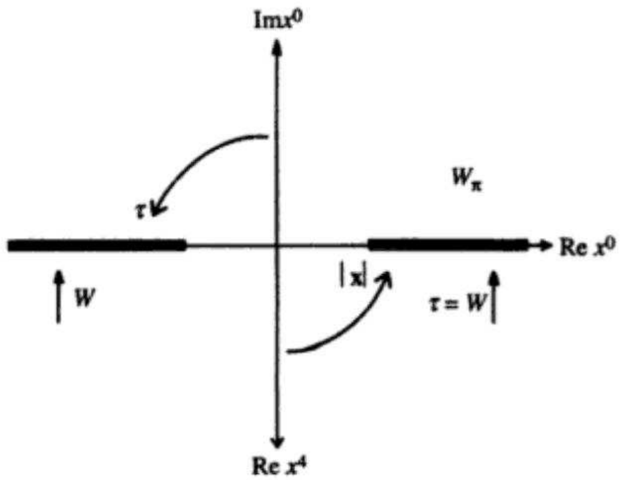
for $x_k \in \mathbb{R}^4$.

Now we introduce the time-ordered Green functions, which will play a fundamental role in quantum field theory, as

$$\tau\left(x_1, \ldots, x_n\right) = \langle 0| T\phi\left(x_1\right) \cdots \phi\left(x_n\right) |0\rangle, \tag{44}$$

where[22] T is the time ordering operator. We see that τ is symmetric in its arguments and that for the 2-point function we have

$$\tau\left(x\right) = \begin{cases} W\left(x\right) & , x^0 > 0 \\ W\left(-x\right), & x^0 < 0. \end{cases} \tag{45}$$



Hence, knowing that $W\left(x\right)$ is analytic in the lower half complex plane and the opposite for $W\left(-x\right)$, we obtain the τ functions by approaching the x^0 real axis in the complex plane through a counter-clockwise rotation. If we generalize this behaviour we find the Wick rotation

$$\tau\left(x_1, \ldots, x_n\right) = \lim_{\phi \to \frac{\pi}{2}} S\left(\ldots; \vec{x}_k, e^{i\phi} x_k^0; \ldots\right). \tag{46}$$

[22]We define the time ordering operator as

$$T\phi\left(x_1\right)\phi\left(x_2\right) = \theta\left(x_1^0 - x_2^0\right)\phi\left(x_1\right)\phi\left(x_2\right) + \theta\left(x_2^0 - x_1^0\right)\phi\left(x_2\right)\phi\left(x_1\right),$$

where θ is the Heaviside step function.

4 Path integral formalism in quantum field theory

In the previous chapter we have dealt with the Euclidean correlation functions and now we can conclude that symmetry property $E2$ in the appendix $(A.2)$ implies that the Euclidean fields commute, this means that they have the same behaviour like the classical fields. Hence, we start this section considering the Euclidean fields not as operators but as random variables, whose expectation is given by

$$\langle F[\phi]\rangle = \int d\mu F[\phi]. \tag{47}$$

We note that the Schwinger functions can fix the measure $d\mu$ uniquely, up to an overall normalization, and then they are called moments of the measure $d\mu$.

The Euclidean functional integral[23] is given by the measure

$$d\mu = \frac{1}{Z} e^{-S[\phi]} \prod_x d\phi(x), \tag{48}$$

where $Z = \int \Pi_x d\phi(x) e^{-S[\phi]}$, and $S[\phi]$ is the Euclidean action. We notice that combining these two equations we get

$$\langle F[\phi]\rangle = \frac{1}{Z} \int \prod_x d\phi(x) F[\phi] e^{-S[\phi]}, \tag{49}$$

and this is the analogous formula of the equation (33) in the Euclidean path integral formalism. In this way it is not necessary to use the operators in the Hilbert space, also for an interacting field. We see that in analogy with statistical mechanics the term $e^{-S[\phi]}$ can be interpreted as the Boltzmann factor.

4.1 Bosonic field theory

In this section we try to find the expression (47) for the Euclidean correlation functions $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ for a real scalar field, which is an example of a bosonic field. We start from the Gaussian integral formalism, using a discretization of the field operator to finite-dimensional vectors and matrices.

[23]This means the Euclidean path integral formalism in quantum field theory.

Using the results shown in Appendix (A.4), we can define the moments of the Gaussian weight function[24] $\exp\left(-\frac{1}{2}(\phi, A\phi)\right)$ as

$$\langle \phi_{i_1} \cdots \phi_{i_n} \rangle \equiv \frac{1}{Z_0} \int d^k\phi \; \phi_{i_1} \cdots \phi_{i_n} \exp\left(-\frac{1}{2}(\phi, A\phi)\right). \tag{50}$$

Hence, with the properties of the Gaussian integrals, we can find these moments by differentiating $Z_0(J)$,

$$\begin{aligned} \langle \phi_{i_1} \cdots \phi_{i_n} \rangle &= \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_n}} Z_0(J)\Big|_{J=0} = \\ &= \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_n}} \exp\left(\frac{1}{2}\left(J, A^{-1}J\right)\right)\Big|_{J=0}, \end{aligned} \tag{51}$$

where[25]

$$Z_0 = \int d^k\phi \exp\left(-\frac{1}{2}(\phi, A\phi)\right) \tag{52}$$

and

$$Z_0(J) = \int d^k\phi \exp\left(-\frac{1}{2}(\phi, A\phi) + (J, \phi)\right). \tag{53}$$

In general we find 0 for the odd moments and for the even moments

$$\langle \phi_{i_1} \cdots \phi_{i_{2n}} \rangle = \sum_{pairings} A^{-1}_{j_1 k_1} A^{-1}_{j_2 k_2} \cdots A^{-1}_{j_n k_n}. \tag{54}$$

Now, before extending this analysis to infinite dimension, we shortly discuss the Euclidean free field. The Euclidean action is given by[26]

$$S_0[\phi] = \int d^4x \left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2\right) = \int d^4x \frac{1}{2}\phi(x)\left(\Box + m^2\right)\phi(x), \tag{55}$$

[24]Notice that this function can be interpreted as the exponential of an action with only the quadratic term.

[25]See the appendix (A.4).

[26]If we move from quantum mechanics to quantum field theory we have to use a relativistic approach. This means that we have the following relation between energy and momentum,

$$E^2 = p^2 + m^2.$$

Using this relation we will not find the Schrödinger equation, but the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi = 0.$$

We remember that this equation can be found with the principle of least action and hence from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\phi(x)\left(\Box - m^2\right)\phi(x),$$

where we define $\Box = -\partial_\mu \partial^\mu$. The propagator

$$G(x,y) = \langle \phi(x)\,\phi(y) \rangle \tag{56}$$

satisfies[27]

$$\left(\Box + m^2\right) G(x,y) = \delta(x-y)\,, \tag{57}$$

then we find

$$G(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{p^2+m^2}. \tag{58}$$

From Wick's Theorem[28] we have

$$G(x_1,\ldots,x_{2n}) \equiv \langle \phi(x_1)\cdots\phi(x_{2n}) \rangle = \sum_{pairings} G(x_{j_1},x_{k_1})\cdots G(x_{j_n},x_{k_n}). \tag{59}$$

Now let $J(x)$ be a classical source, then we have

$$(J,\phi) = \int d^4x J(x)\,\phi(x), \tag{60}$$

and we can define the generating functional of Green's functions as

$$Z[J] \equiv \left\langle e^{(J,\phi)} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n J(x_1)\cdots J(x_n)\, G(x_1,\ldots,x_n), \tag{61}$$

that is normalized to $Z[0]=1$. Then it follows[29]

$$G(x_1,\ldots,x_n) = \left. \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)} \right|_{J=0}. \tag{62}$$

---

and the action

$$S = \int L dt = \int \left(\int \mathcal{L} d^3x \right) dt,$$

we find, through $\delta\mathcal{L}=0$, the Klein-Gordon equation.

Then, in the Euclidean space-time, we have the action

$$S_E = \int d^4x \frac{1}{2} \phi(x) \left(\Box + m^2\right) \phi(x).$$

[27]We see that, if the mass term m vanishes, we have a propagation like a wave. This is the case of the electromagnetic field.

[28]See the Appendix ($A.3$).

[29]We have to introduce the functional derivative $\frac{\delta}{\delta J(y)} F[J]$ through

$$\lim_{\epsilon\to\infty} \frac{1}{\epsilon} \left(F[J+\epsilon h] - F[J]\right) = \int d^4x\ h(x) \frac{\delta}{\delta J(x)} F[x].$$

We note that for free field we have

$$Z[J] \equiv Z_0[J] = \exp\left(\frac{1}{2}\int d^4x d^4y J(x) G(x,y) J(y)\right) = \\ = \exp\left(\frac{1}{2}(J, G_0 J)\right). \quad (63)$$

At this point we can return to Euclidean formalism. If we compare the generating functional $Z_0[J]$ between the Gaussian integrals and the Green functions we see that we go from a discrete to a continuous representation, this means

$$\phi_i \rightarrow \phi(x), \quad (64)$$

$$\frac{\partial}{\partial J_i} \rightarrow \frac{\delta}{\delta J(x)}. \quad (65)$$

Then if we rewrite the equation (63) as $Z_0[J] = \exp\left(\frac{1}{2}(J, GJ)\right)$, and compare with the Gaussian integral formalism[30], we find

$$A^{-1} \rightarrow G, \quad (66)$$

and conversely

$$A \rightarrow G^{-1}. \quad (67)$$

From equation (57) we find $G^{-1} = \Box + m^2$, and then it follows

$$\frac{1}{2}(\phi, A\phi) \rightarrow \frac{1}{2}\left(\phi, G^{-1}\phi\right) = \frac{1}{2}\left(\phi, \left(\Box + m^2\right)\phi\right) = \\ = \frac{1}{2}\int d^4x \phi(x)\left(\Box + m^2\right)\phi(x) = S_0[\phi]. \quad (68)$$

Now, using the results shown in the Appendix (A.4) and the last equation, we find

$$Z_0[J] = \frac{1}{Z_0}\int \prod_x d\phi(x) \exp\left(-\frac{1}{2}\left(\phi, \left(\Box + m^2\right)\phi\right) + (J, \phi)\right) = \\ = \frac{1}{Z_0}\int \prod_x d\phi(x) \exp(-S_0 + (J, \phi)), \quad (69)$$

where

$$Z_0 = \int \prod_x d\phi(x) \exp\left(-\frac{1}{2}\left(\phi, \left(\Box + m^2\right)\phi\right)\right). \quad (70)$$

[30] See the Appendix (A.4).

The difference between this infinite-dimensional integral and the Gaussian integral is that the first is not well-defined, but we can solve this problem defining the measure

$$d\mu_0(\phi) = \frac{1}{Z_0} D[\phi]\, e^{-S_0(\phi)}, \tag{71}$$

where

$$D[\phi] \equiv \prod_x d\phi(x). \tag{72}$$

We notice that we have found exactly the equation (48), this means the measure for the Euclidean path integral formalism.

We remember that the generating functional $Z_0[J]$ is connected to the Green functions, so we can conclude that for a free field we have

$$\begin{aligned} \langle \phi(x_1) \cdots \phi(x_n) \rangle &= \int d\mu_0(\phi)\, \phi(x_1) \cdots \phi(x_n) = \\ &= \frac{1}{Z_0} \int \prod_x d\phi(x)\, e^{-S_0[\phi]} \phi(x_1) \cdots \phi(x_n). \end{aligned} \tag{73}$$

4.2 Interacting field

Before showing the connection with the perturbation theory, it is interesting to see the behaviour of the functional integrals for an interacting field. In our approach we neglect the problems associated with divergences and renormalization. We use the following Euclidean action

$$S[\phi] = S_0[\phi] + S_I[\phi], \tag{74}$$

where S_I describes the interaction part. Rewriting Dyson's Formula[31] in terms of the generating functionals we find

$$Z[J] = \frac{\langle 0 | \exp(-S_I[\phi_{in}] + (J, \phi_{in})) | 0 \rangle}{\langle 0 | \exp(-S_I[\phi_{in}]) | 0 \rangle}, \tag{75}$$

where ϕ_{in} is the free field, this means without interaction. Comparing to the Gauss functional integral we find

$$Z[J] = \frac{1}{Z} \int \prod_x d\phi_{in}(x)\, e^{-S[\phi] + (J,\phi)} \tag{76}$$

[31] Dyson's formula for the correlation functions in Euclidean space is given by

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\langle 0 | \phi_{in}(x_1) \cdots \phi_{in}(x_n) \exp(-S_I[\phi_{in}]) | 0 \rangle}{\langle 0 | \exp(-S_I[\phi_{in}]) | 0 \rangle}.$$

where

$$Z = \int \prod_x d\phi_{in}(x)\, e^{-S[\phi]}, \tag{77}$$

and the correlation functions are given by

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{Z} \int \prod_x d\phi_{in}(x)\, e^{-S[\phi]} \phi_{in}(x_1) \cdots \phi_{in}(x_n). \tag{78}$$

5 Connection with perturbative expansion

In this last chapter we will see that the rules of perturbation theory, this means the Feynman rules, are deduced from the functional integrals. We begin by considering a scalar field with a quartic self-interaction given by

$$S_I[\phi] = \frac{g}{4!} \int d^4x \phi(x)^4. \tag{79}$$

Then, from equation (78) and expanding the exponential of the interaction we find the following n-point function

$$G(x_1, \ldots, x_n) = \frac{1}{Z} \int \prod_x d\phi(x)\, e^{-S_0[\phi]} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{g}{4!} \int d^4x \phi(x)^4 \right)^n \phi(x_1) \cdots \phi(x_n). \tag{80}$$

Now, using the equation (76), we can write the generating functional as

$$\begin{aligned}
Z[J] &= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_x d\phi(x) \left(-\frac{g}{4!} \int d^4x \phi(x)^4 \right)^n e^{-S_0[\phi] + (J,\phi)} = \\
&= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{g}{4!} \int d^4x \left(\frac{\delta}{\delta J(x)} \right)^4 \right)^n \int \prod_x d\phi(x)\, e^{-S_0[\phi] + (J,\phi)} = \\
&= \frac{1}{Z} \exp\left(-\frac{g}{4!} \int d^4x \left(\frac{\delta}{\delta J(x)} \right)^4 \right) Z_0 Z_0[J] = \\
&= \frac{Z_0}{Z} \exp\left(S_I\left[\frac{\delta}{\delta J(x)} \right] \right) \exp\left(\frac{1}{2}(J, G_0 J) \right).
\end{aligned} \tag{81}$$

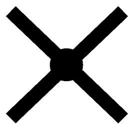
In the same way we can find that the Green functions are given by

$$G(x_1, \ldots, x_n) = \frac{Z_0}{Z} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \exp\left(S_I\left[\frac{\delta}{\delta J(x)} \right] \right) \exp\left(\frac{1}{2}(J, G_0 J) \right) \Bigg|_{J=0}. \tag{82}$$

We can interpret graphically these Green functions. We begin by expanding the exponential of $S_I\left[\frac{\delta}{\delta J(x)}\right]$, then:

- each derivative $\frac{\delta}{\delta J(x_i)}$ is indicated by an external point from which a line emerges (external scalar term)

- each factor $-g \int d^4x \left(\frac{\delta}{\delta J(x_i)}\right)^4$ is indicated by an internal vertex, from which four lines emerges. (vertex term)



We can also expand $\exp\left(\frac{1}{2}\left(J, G_0 J\right)\right)$, and this term is indicated by the internal lines. (propagator term)

Till now we haven't regarded the term $\frac{Z_0}{Z}$ in the Green functions, but this allows us to normalize the generating functional to $Z\left(0\right) = 1$, using

$$\frac{Z}{Z_0} = \exp\left(-S_I\left[\frac{\delta}{\delta J\left(x\right)}\right]\right)\exp\left(\frac{1}{2}\left(J, G_0 J\right)\right)\Bigg|_{J=0}. \tag{83}$$

In a graphical representation of this equation, we haven't external points and these graphs are called vacuum graphs. Using this normalization these terms cancel out in the graphical representation of Green's functions. Even though we are using this normalization we can find divergences, that come from the integrals over internal loops. In fact we cancel out only the vacuum graphs and not all the internal loops. This means that the $Z\left[J\right]$ functional is not well-defined. For example, we have some terms proportional to

$$g \int d^4p \, \frac{1}{p^2 + m^2}, \tag{84}$$

which diverge. We need to find a method to evaluate this integral, separating the divergent part and in perturbation theory it is possible to find different regularizations. For example we can use dimensional regularization, where we evaluate the integral (84) in other dimensions, or a lattice space, which implies that the momenta are periodic and hence bounded. These regularization methods allow one to identify the divergent contributions in the loop integrals; then, such divergences can be formally "reabsorbed" by a redefinition of the parameters of the theory: this is called "renormalization", and the "renormalized" values of the parameters are considered as the physical ones (to be compared with the experiments). The field theories for which, at every order in perturbation theory, only a finite number of parameters needs to be renormalized are called "renormalizable theories".

A Appendix

A.1 Main ingredients of quantum field theory for a real scalar field in Minkowski space

- A1: There is a Hilbert space $\mathcal{H}$ of physical states, containing a vacuum state $|0\rangle$.

- A2: On $\mathcal{H}$ we have aunitary representation $U(a, \Lambda)$ of the Poincaré group, where a and Λ denote a spacetime translation and a rotation/boost, respectively. The vacuum is invariant under these transformations.

- A3 (Spectrum condition): The generators P^μ of translations,

$$U(a, 1) = \exp\left(iP_\mu a^\mu\right),$$

have a spectrum, which is contained in the forward light cone

$$\overline{V}_+ = \left\{q \in \mathbb{R}^4 : q^0 \geq 0, q^\mu q_\mu \geq 0\right\}.$$

$P^0 \equiv H$ is the Hamiltonian.

- A4: The vacuum is the only vector invariant under $U(a, \Lambda)$.

- F1: We have a field $\phi(x)$ acting as an operator on $\mathcal{H}$.

- F2: The field transforms covariantly:

$$U(a, \Lambda)\phi(x)U^{-1}(a, \Lambda) = \phi(\Lambda x + a)$$

- F3 (locality): The field commutes for space-like separations:

$$[\phi(x), \phi(y)] = 0,$$

for $(x - y)^2 \leq 0$.

A.2 Properties of the Schwinger functions

- E1 (Euclidean covariance): The Schwinger functions are covariant under Euclidean transformations

$$S\left(x_1, \ldots, x_n\right) = S\left(\Lambda x_1 + a, \quad \ldots, \Lambda x_n + a\right),$$

where $\Lambda \in SO(4)$.

- E2 (reflection positivity): Let be

$$\theta\left(\vec{x}, x^4\right) = \left(\vec{x}, -x^4\right),$$
$$\Theta\phi(x) = \overline{\phi(\theta x)}$$

the Euclidean time reflection[32], where $\overline{\phi(\theta x)}$ is the complex conjugation of $\phi(\theta x)$, and F a function of the fields at positive times, then

$$\langle (\Theta F) F \rangle \geq 0.$$

- E3 (symmetry): Schwinger functions are symmetric in their arguments.

A.3 Wick's Theorem

This section has not the purpose to show the Wick's Theorem, but we want only to explain how we can find equation (59). We start to evaluate $\langle 0| T\phi(x_1)\phi(x_2)|0\rangle$ and then we will generalize to more field operators. The field operator $\phi(x)$ can be decompose in two terms in the following way,

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

where

$$\phi^+(x)|0\rangle = 0, \ \langle 0|\phi^-(x) = 0.$$

If we set $x^0 > y^0$, we find

$$\begin{aligned} T\phi(x)\phi(y) &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) = \\ &= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \\ &\quad \left[\phi^+(x), \phi^-(y)\right], \end{aligned}$$

where we have rewritten the terms in the normal order, in this way these will vanish for the vacuum expectation value. Now, if we extend to $x^0 < y^0$, we have to replace the commutator with the following contraction,

$$\widehat{\phi(x)\phi(y)} \equiv \begin{cases} \left[\phi^+(x), \phi^-(y)\right], \ x^0 > y^0 \\ \left[\phi^+(y), \phi^-(x)\right], \ y^0 > x^0 \end{cases}.$$

[32] Θ is the Euclidean equivalent of Hermitian conjugation in Minkowski space. We need a time reflection, because we have a change in the sign of t for the Hermitian conjugation of the time evolution operator $\exp(-iHt)$.

If we use the explicit definitions

$$\begin{aligned}\phi^{+}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_p e^{-ip\cdot x},\\ \phi^{-}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}{a_p}^{\dagger} e^{+ip\cdot x},\end{aligned}$$

we can find the following relation between this contraction and the Feynman propagator

$$G(x,y) = \widehat{\phi(x)\,\phi(y)}.$$

We generalize these relations writing

$$T\phi(x_1)\cdots\phi(x_n) = N\left\{\phi(x_1)\cdots\phi(x_n) + \textit{all possible contractions}\right\}.$$

Evaluating this chain of operators in the vacuum we find that the only non vanishing terms are those with the maximal number of contractions and knowing the relation between a contraction and the Feynman propagator we have

$$G(x_1,\ldots,x_{2n}) = \sum_{pairings} G(x_{j_1},x_{k_1})\cdots G(x_{j_n},x_{k_n}).$$

A.4 Properties of the Gaussian integrals

- $$\int_{-\infty}^{\infty} d\phi \exp\left(-\frac{1}{2}A\phi^2\right) = \left(\frac{2\pi}{A}\right)^{\frac{1}{2}},$$

  for $Re(A) > 0$.

- $$Z_0 \equiv \int d^k\phi \exp\left(-\frac{1}{2}(\phi, A\phi)\right) = (2\pi)^{\frac{k}{2}}(\det A)^{-\frac{1}{2}},$$

  where $A = (A_{ij})$ is a real, symmetric positive matrix and with $(\phi, A\phi) = \phi_i A_{ij}\phi_j$.

- $$Z_0(J) \equiv \frac{1}{Z_0}\int d^k\phi \exp\left(-\frac{1}{2}(\phi, A\phi) + (J,\phi)\right) = \exp\left(\frac{1}{2}\left(J, A^{-1}J\right)\right),$$

  where J is an arbitrary vector. This equation can be obtained by completing the square in the Gaussian integral, and by a shift in the integration variables.

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