Entanglement between system and environment II

Proseminar FS09: Information in Statistical Mechanics

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Context

- Entanglement between system and environment I: Averaging over possible quantum states yields thermodynamics
- *Entanglement between system and environment II: How can thermodynamics be derived without averaging and the a priori assumption of equipartition?*
- Entanglement between system and environment III: How is thermal equilibrium reached?

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Assumptions used up to now

- states are a priori equally probable
- interaction between system and environment are small
- the environment has an appropriate spacing of the energy levels

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Ensembles of Statistical Physics

$$\Omega_S^{(\mu c)} = \frac{1}{d_S} |m\rangle \langle m| = \frac{I_S}{d_S}$$

$$\Omega_S^{(can.)} = \frac{e^{-\frac{E_m}{k_B T}}}{\sum_n e^{-\frac{E_n}{k_B T}}} |m\rangle \langle m| = \frac{e^{-\frac{H}{k_B T}}}{\mathrm{Tr} e^{-\frac{H}{k_B T}}}$$

$$\Omega_S^{(gc)} = \frac{e^{-\frac{H-\mu \widehat{N}}{k_B T}}}{\mathrm{Tr} e^{-\frac{H-\mu \widehat{N}}{k_B T}}}$$

where $T : \mathrm{Tr}\left(\Omega_S^{(can.)} H\right) = E, \quad \mu : \mathrm{Tr}\left(\Omega_S^{(gc)} \widehat{N}\right) = N$

and the von Neumann Entropy: $-\mathrm{Tr}\,(\Omega \log \Omega)$

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Hilbert Space and reduction to the system S

- $\mathcal{H}_R \subseteq \mathcal{H}_S \otimes \mathcal{H}_E$ $\qquad d_R \leq d_S \cdot d_E$
- arbitrary restriction R

old:

- equiprobable (maximally mixed) state: $\mathscr{E}_R = \frac{I_R}{d_R}$
- canonical state of system S relative to restriction R: $\Omega_S = \mathrm{Tr}_E \mathscr{E}_R$

new:

- pure state $|\varphi\rangle \in \mathcal{H}_R$, chosen randomly according to the (unitarily invariant) Haar measure
- reduced state of system S: $\rho_S = \mathrm{Tr}_E |\varphi\rangle \langle\varphi|$

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Distances, Norms & Averages

The *trace distance*

$$D(\rho_S, \Omega_S) = ||\rho_S - \Omega_S||_1 = \frac{1}{2}\mathrm{Tr}\sqrt{(\rho_S - \Omega_S)^\dagger (\rho_S - \Omega_S)}$$
$$= \sup_{||O|| \leq 1} \mathrm{Tr}\left((\rho_S - \Omega_S) O\right)$$

is equal to the maximal difference between two states in the probability of obtaining any measurement outcome.

Hilbert-Schmidt norm

$||\rho_S - \Omega_S||_2 = \sqrt{\mathrm{Tr}(\rho_S - \Omega_S)^\dagger (\rho_S - \Omega_S)}$

$\forall M \in \mathbb{C}^{n \times n} \ \ ||M||_1^2 \leq n||M||_2^2$

$\langle \cdot \rangle$ average over all pure states $|\varphi\rangle \in \mathcal{H}_R$ according to the Haar measure, e.g. $\Omega_S = \langle \rho_s \rangle$

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Principle of apparently a priori equal Probability

For almost every pure state $|\varphi\rangle \in \mathcal{H}_R$, the system S behaves as if the universe were in the equiprobable state $\mathscr{E}_R$,
i.e. for almost every pure state of the universe is locally (that is, on the system) indistinguishable from $\mathscr{E}_R$.
This means that $\rho_S \approx \Omega_S$.

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Grand Canonical Principle

Given a sufficiently small subsystem of the universe, almost every pure state of the universe is such that the subsystem is approximately in the canonical state Ω_S.

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Thermal Canonical Principle

Given that the total energy of the universe is approximately E, interactions between the system and the rest of the universe are weak, and that the density of states of the environment increases approximately exponentially with energy, almost every pure state of the universe is such that the state of the system alone is approximately equal to the thermal canonical state $\propto e^{-\frac{H}{k_B T}}$, with temperature T (corresponding to the energy E).

Theorem

For a randomly chosen state $|\varphi\rangle \in \mathcal{H}_R$ and $\forall \varepsilon > 0$ the trace distance between the reduced density matrix ρ_S and the canonical state Ω_S is given probabilistically by

$$\text{Prob}\left[D\left(\rho_S\left(\varphi\right), \Omega_S\right) \geq \eta\right] = \leq \eta',$$

where $\eta = \varepsilon + \frac{1}{2}\sqrt{\frac{d_S}{d_E^{eff}}} \leq \varepsilon + \frac{1}{2}\sqrt{\frac{d_S^2}{d_R}}$ and

$\eta' = 4exp\left(-\frac{2}{9\pi^3} d_R \varepsilon^2\right)$;

the effective size of the environment is $d_E^{eff} = \frac{1}{\text{Tr}\Omega_E^2} \geq \frac{d_R}{d_S}$

and $\Omega_E = \text{Tr}\mathscr{E}_R = \langle \rho_E \rangle$.

When $d_R \gg 1$ and $d_E^{eff} \gg d_S$, we set $\varepsilon = d_R^{-1/3}$, then η and η' become small.

Entanglement between system and environment II

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Levy's Lemma

For any function on a high dimensional hypersphere the overwelming majority of values are close to the average.

$\forall f : S_d \to \mathbb{R}, \varphi \in S_d$ chosen uniformly at random, $\forall \varepsilon > 0$
$\text{Prob}\left[|f(\varphi) - \langle f \rangle| \geq \varepsilon\right] \leq 2exp\left(-\frac{d+1}{9\pi^3\eta^2}\varepsilon^2\right)$,
where the Lipschitz constant $\eta = \sup |\nabla f|$.

Pure states in Hilbert spaces can be represented as points on hyperspheres, in our case $\mathcal{H}_R \cong S_{2d_R-1}$.

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Proof

Applying Levy's Lemma on

$$f(\varphi) = D\left(\rho_S(\varphi), \Omega_S\right) = ||\rho_S(\varphi) - \Omega_S||_1$$

and noting that $f(\varphi)$ has Lipschitz constant $\eta \leq 2$, we get

$$\text{Prob}\left[|||\rho_S - \Omega_S||_1 - \langle||\rho_S - \Omega_S||_1\rangle| \geq \varepsilon\right] \leq 2e^{-\frac{2d_R}{9\pi^3}\varepsilon^2}.$$

Rearranging gives

$$\text{Prob}\left[||\rho_S - \Omega_S||_1 \geq \eta\right] \leq \eta'$$

where $\eta = \varepsilon + \langle||\rho_S - \Omega_S||_1\rangle$ and $\eta' = 4exp\left(-\frac{2}{9\pi^3}d_R\varepsilon^2\right)$. It remains to show that

$$\langle||\rho_S - \Omega_S||_1\rangle \overset{!}{\leq} \frac{1}{2}\sqrt{\frac{d_S}{d_E^{eff}}}.$$

Measurement Operators

Applying Levy's Lemma on the expectation value of a measurement O:

$$f(\varphi) = \mathrm{Tr}\left(O_S \rho_S\right)$$

and noting that $f(\varphi)$ has Lipschitz constant $\eta \leq 2\left|\left|O_S\right|\right|$, we get

$$\mathrm{Prob}\left[\left|\mathrm{Tr}\left(O_S\rho_S\right) - \left\langle \mathrm{Tr}\left(O_S\rho_S\right)\right\rangle\right| \geq \varepsilon\right] \leq 2e^{-\frac{2d_R}{9\pi^3\left|\left|O_S\right|\right|^2}\varepsilon^2}.$$

However

$$\left\langle \mathrm{Tr}\left(O_S\rho_S\right)\right\rangle = \mathrm{Tr}\left(O_S\left\langle\rho_S\right\rangle\right) = \mathrm{Tr}\left(O_S\Omega_S\right)$$

and choosing $\varepsilon = d_R^{-1/3}$ we obtain

$$\mathrm{Prob}\left[\left|\mathrm{Tr}\left(O_S\rho_S\right) - \mathrm{Tr}\left(O_S\Omega_S\right)\right| \geq d_R^{-1/3}\right] \leq 2e^{-\frac{2d_R^{1/3}}{9\pi^3\left|\left|O_S\right|\right|^2}}.$$

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Operator Basis

Complete orthogonal operator basis for $\mathcal{H}_S$:

$U_S^0, \ldots, U_S^{d_S^2-1}$

$||U_S^x|| = 1 \quad \forall 0 = 1, \ldots, d_S^2 - 1$

$$\text{Prob}\left[\exists x : \left|\text{Tr}\left(U_S^x \rho_S\right) - \text{Tr}\left(U_S^x \Omega_S\right)\right| \geq d_R^{-1/3}\right]$$

$$\leq 2d_S^2 e^{-\frac{2d_R^{1/3}}{9\pi^3}}$$

It is very likely that all operators U_S^x will have expectation values close to their canonical values.

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Bound on $D(\rho_S, \Omega_S)$ (I)

Using the complete basis of U_S^x to expand ρ_S

$$\rho_S = \frac{1}{d_S} \sum_{x=0}^{d_S^2-1} \mathrm{Tr}\left(U_S^{x\dagger} \rho_S\right) U_S^x \equiv \frac{1}{d_S} \sum_x C_x(\rho_S)\, U_S^x$$

we obtain

$$\mathrm{Prob}\left[\exists x : |C_x(\rho_S) - C_x(\Omega_S)| \geq \varepsilon\right] \leq 2 d_S^2 e^{-\frac{2d_R}{9\pi^3}\varepsilon^2}.$$

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Bound on $D(\rho_S, \Omega_S)$ (II)

when $|C_x(\rho_S) - C_x(\Omega_S)| \leq \varepsilon \qquad \forall x$

$$||\rho_S - \Omega_S||_2^2 = \left|\left|\frac{1}{d_S}\sum_x \left(C_x(\rho_S) - C_x(\Omega_S)\right) U_S^x\right|\right|_2^2$$

$$= \frac{1}{d_S^2}\mathrm{Tr}\left(\sum_x \left(C_x(\rho_S) - C_x(\Omega_S)\right) U_S^x\right)^2$$

$$= \frac{1}{d_S}\sum_{x=0}^{d_S^2-1}\left(C_x(\rho_S) - C_x(\Omega_S)\right)^2 \leq d_S \varepsilon^2$$

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Bound on $D(\rho_S, \Omega_S)$ (III)

Using the bound of the trace-norm by the Hilbert-Schmidt-norm

$$D(\rho_S, \Omega_S) = ||\rho_S - \Omega_S||_1 \leq \sqrt{d_S}\, ||\rho_S - \Omega_S||_2 \leq d_S \varepsilon$$

and choosing $\varepsilon = \left(\frac{d_S}{d_R}\right)^{1/3}$ we obtain

$$\mathrm{Prob}\left[D(\rho_S, \Omega_S) \geq \left(\frac{d_S^2}{d_R}\right)^{1/3}\right] \leq 2 d_S^2 e^{-\frac{2 d_R^{1/3}}{9\pi^3}}.$$

Finally for $d_R \gg d_S^2 \gg 1$ we get therefore $D(\rho_S, \Omega_S) \approx 0$.

Spin-1/2 System with known Energy

k spins in system S, $n-k$ spins in the environment, constant external field B

$$H = -\sum_{i=1}^{n} \frac{B}{2} \sigma_z^{(i)}$$

All states with the same total number of spins aligned with the field np form subspace $\mathcal{H}_R$. If $np > k$ then $d_S = 2^k$ and

$$d_R = \binom{n}{np}.$$

Further calculations ob the board...

Main Results

- averages are not necessary
- Principle of Apparantly Equal a priory Probabilities: *Almost every pure state of the universe is locally indistinguishable from the canonical state.*
- no limitation to weak interactions with the environment
- more general restrictions on the universe can be treated

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Thank you for your attention.

What questions do you have?

After the break Daniel will continue our Series on *Entanglement between system and environment*

Literature

- Sandu Popescu, Anthony Short & Andreas Winter: *Entanglement and the foundations of statistical mechanics*,
  nature physics, Nov 2006, Vol. 2, p. 754-758
- Sandu Popescu, Anthony Short & Andreas Winter: *The foundations of statistical mechanics from entanglement: Individual states vs. averages*, arXiv:quant-ph/0511225v3, 17 Oct 2006

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