

# Information erasure in a stochastic model

Mar 30, 2009

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Prove Landauer's claim for the case of a specific physical application  
without presuming the second law of thermodynamics

paper:

**Heat generation required by information erasure**

Kousuke Shizume, 1995

# Overview

## 1. Introduction:

Landauer's claim (revisited), stochastic model

## 2. Stochastic movement:

Random walk, Focker-Planck equation

## 3. Computation of the minimal heat generation

## 4. Discussion

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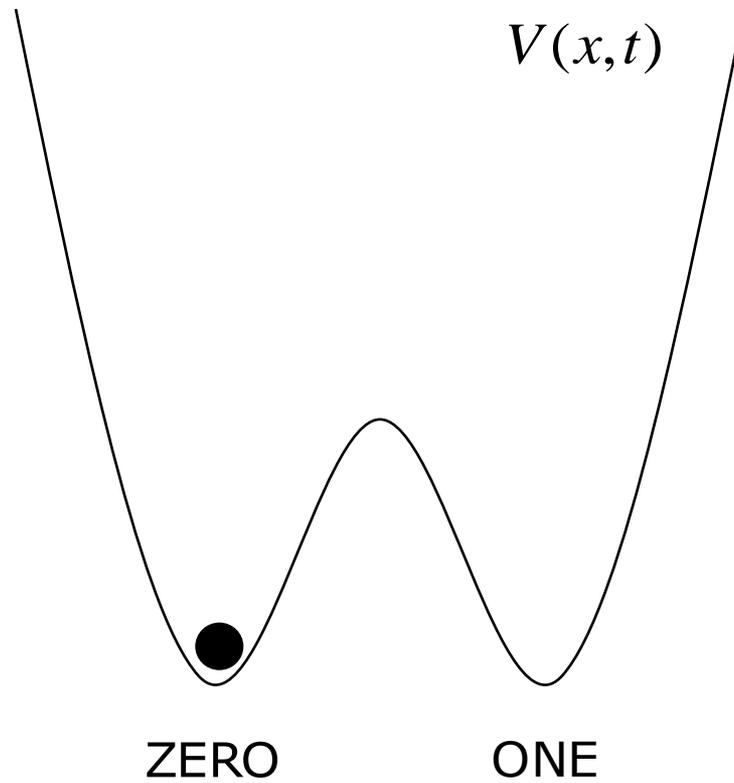
# Landauer's claim

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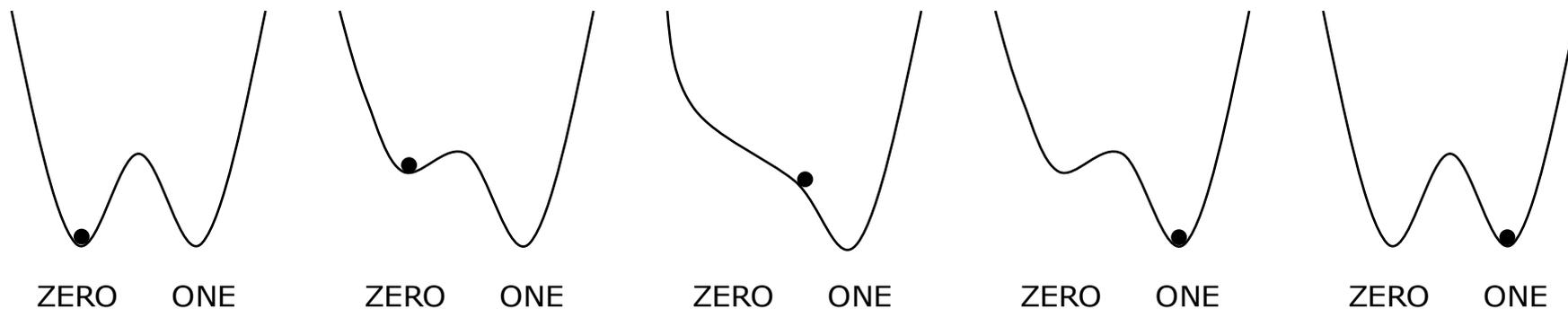
"erasure of 1 bit of information requires a minimal heat generation of  $k_B T \ln 2$ "

→ physical system with two stable states and an operation RESTORE TO ONE (RTO)

# The bistable potential



# Operation RESTORE TO ONE (RTO)



This procedure is independent of the initial value of the memory

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- bistable potential need dissipative system

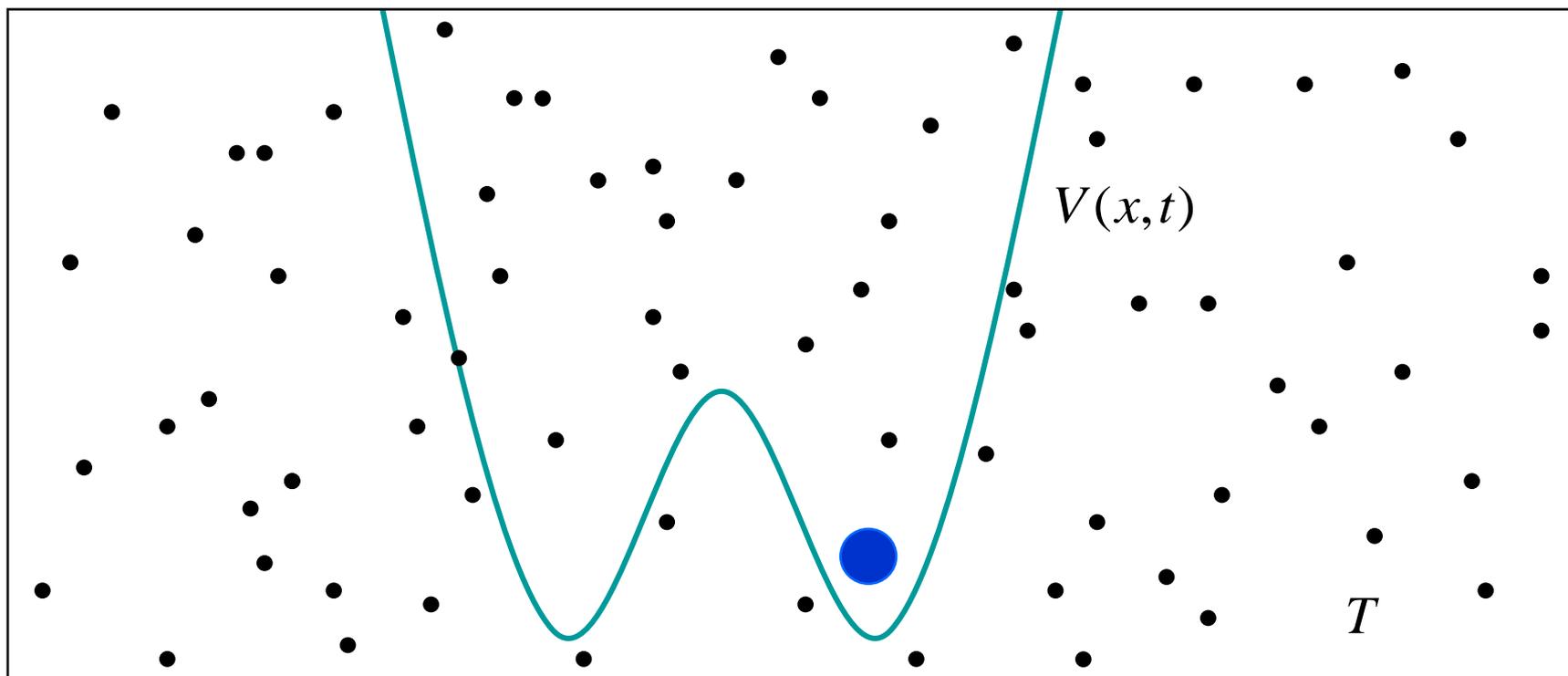
# Landauer's claim

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"erasure of 1 bit of information requires a minimal heat generation of  $k_B T \ln 2$ "

- physical system with two stable states and an operation RESTORE TO ONE (RTO)
- bistable potential: need dissipative system
- particular memory device: particle doing brownian motion in an additional (bistable) potential

## Stochastic model for information erasure



$$m \frac{d^2 x}{dt^2} + m\gamma \frac{dx}{dt} = -\frac{\partial V(x,t)}{\partial x} + F_R(t): \quad \text{Langevinequation}$$

$F_R(t)$ : randomforce: time dependentstochasticvariable

# Brownian motion

Langevinequation:

$$m \frac{d^2 x}{dt^2} + m\gamma \frac{dx}{dt} = -\frac{\partial V(x,t)}{\partial x} + F_R(t)$$

$F_R$  : Gaussianwhiterandomforce

$$\langle F_R(t_1)F_R(t_2) \rangle = 2m\gamma T \delta(t_1 - t_2)$$

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Fokker-Planck equation(FPE):

$$\frac{\partial}{\partial t} f(x,u,t) = \left[ -\frac{\partial}{\partial x} u + \frac{\partial}{\partial u} \left( \gamma u + \frac{1}{m} \frac{\partial V(x,t)}{\partial x} \right) + \frac{\gamma T}{m} \frac{\partial^2}{\partial u^2} \right] f(x,u,t)$$

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## 2. Stochastic movement:

- Time dependent stochastic variables
- Markov processes
- Example: random walk
- The Focker-Planck equation

## 3. Computation of the minimal heat generation

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## Time dependent stochastic variables

$P(y_1, t_1)$  probability that  $Y$  takes at  $t = t_1$  the value  $y_1$

$P(y_1, t_1; y_2, t_2; \dots; y_n, t_n)$  probability that  $Y$  takes the values  $y_i$  at  $t = t_i$  for  $i = 1, \dots, n$

$$\int P(y_1, t_1; y_2, t_2; \dots; y_n, t_n) dy_n = P(y_1, t_1; y_2, t_2; \dots; y_{n-1}, t_{n-1})$$

$P(y_1, t_1 | y_2, t_2)$  defined by  $P(y_1, t_1)P(y_1, t_1 | y_2, t_2) = P(y_1, t_1; y_2, t_2)$

$$\Rightarrow P(y_2, t_2) = \int P(y_1, t_1)P(y_1, t_1 | y_2, t_2) dy_1$$

# Markov processes

$$P(y_1, t_1; \dots; y_{n-1}, t_{n-1} | y_n, t_n) = P(y_{n-1}, t_{n-1} | y_n, t_n)$$

Joint probabilities are determined by  $P(y_1, t_1 | y_2, t_2)$  and  $P(y, t)$

$$\begin{aligned} \text{For example, } P(y_1, t_1; y_2, t_2; y_3, t_3) &= P(y_1, t_1; y_2, t_2) P(y_1, t_1; y_2, t_2 | y_3, t_3) \\ &= P(y_1, t_1) P(y_1, t_1 | y_2, t_2) P(y_2, t_2 | y_3, t_3) \end{aligned}$$

# Markov processes

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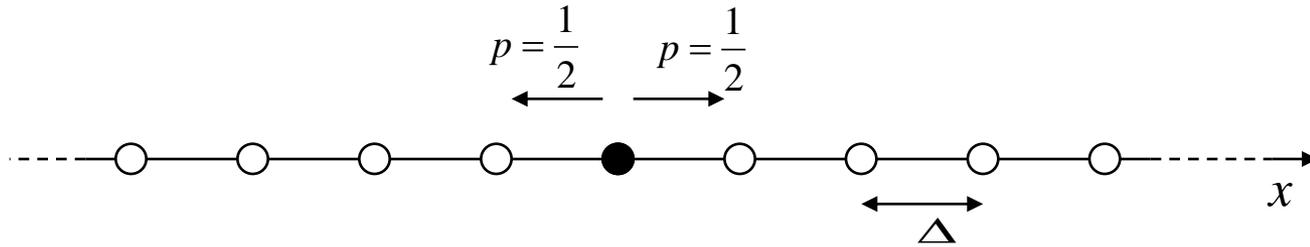
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$$\Rightarrow P(y_1, t_1; y_3, t_3) = P(y_1, t_1) \int P(y_1, t_1 | y_2, t_2) P(y_2, t_2 | y_3, t_3) dy_2$$

$$\Rightarrow P(y_1, t_1 | y_3, t_3) = \int P(y_1, t_1 | y_2, t_2) P(y_2, t_2 | y_3, t_3) dy_2$$

(ChampmanKolmogorovequation)

## Random walk

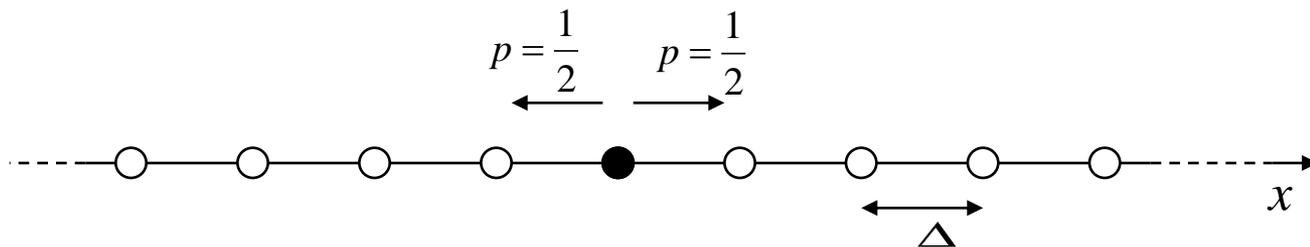


Random variable  $X$  has possible values  $n\Delta$ ,  $n \in \mathbb{Z}$   
 transitions can occur at times  $t = s\tau$ ,  $s \in \mathbb{N}_{\geq 0}$

$$P(y_2, t_2) = \int P(y_1, t_1) P(y_1, t_1 | y_2, t_2) dy_1$$

$$\longrightarrow P(n\Delta, (s+1)\tau) = \sum_{m=-\infty}^{\infty} P(m\Delta, s\tau) P(m\Delta, s\tau | n\Delta, (s+1)\tau)$$

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$$P(m\Delta, s\tau | n\Delta, (s+1)\tau) = \frac{1}{2} \delta_{n(m+1)} + \frac{1}{2} \delta_{n(m-1)}$$

$$\Rightarrow P(n\Delta, (s+1)\tau) = \frac{1}{2} P((n+1)\Delta, s\tau) + \frac{1}{2} P((n-1)\Delta, s\tau)$$

## Random walk

$$P(n\Delta, (s+1)\tau) = \frac{1}{2} P((n+1)\Delta, s\tau) + \frac{1}{2} P((n-1)\Delta, s\tau)$$

take  $\Delta \rightarrow 0$ ,  $\tau \rightarrow 0$  such that  $D \equiv \frac{\Delta^2}{2\tau}$  remains constant

$$\Rightarrow \frac{\partial P(x,t)}{\partial t} = D \frac{\partial^2 P(x,t)}{\partial x^2}$$

$$P(x,0) = \delta(x)$$

$$\Rightarrow P(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$$

# The Fokker-Planck equation

integration of the Langevinequationyields

$$u(t) = u_0 e^{-(\gamma/m)t} + \frac{1}{m} \int_0^t ds e^{-(\gamma/m)(t-s)} F_R(s)$$

$$x(t) = x_0 + \frac{m}{\gamma} (1 - e^{-(\gamma/m)t}) u_0 + \frac{1}{\gamma} \int_0^t ds (1 - e^{-(\gamma/m)(t-s)}) F_R(s)$$

$u(t)$  and  $x(t)$  are stochasticvariables that depend on  $F_R(t)$  in a nontrivial way.

# The Fokker-Planck equation

- assumption: force acting on particle is always finite:

$$\frac{\partial}{\partial t} \int_V f(x, u, t) dx du = - \int_{\partial V} f(x, u, t) \dot{\mathbf{X}} \cdot d\mathbf{\Lambda} = - \int_V \nabla \cdot (f(x, u, t) \dot{\mathbf{X}})$$

$$\Rightarrow \frac{\partial f}{\partial t} = - \nabla \cdot (\dot{\mathbf{X}} f) = - \frac{\partial (x f)}{\partial x} - \frac{\partial (u f)}{\partial u}$$

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- ▶  $\dot{u} = \frac{du}{dt} = -\frac{\gamma}{m} u(t) - \frac{1}{m} \frac{\partial V(x, t)}{\partial x} + \frac{1}{m} F_R(t)$

$$\Rightarrow \frac{\partial f}{\partial t} = - \frac{\partial(uf)}{\partial x} + \frac{\gamma}{m} \frac{\partial(uf)}{\partial u} + \frac{1}{m} \frac{\partial V(x, t)}{\partial x} \frac{\partial f}{\partial u} - \frac{1}{m} F_R \frac{\partial f}{\partial u}$$

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$$\Rightarrow \frac{\partial f}{\partial t} = - \frac{\partial (u f)}{\partial x} + \frac{\gamma}{m} \frac{\partial (u f)}{\partial u} + \frac{1}{m} \frac{\partial V(x, t)}{\partial x} \frac{\partial f}{\partial u} - \frac{1}{m} F_R \frac{\partial f}{\partial u}$$

gives us evolution of probability density for given

$F_R(s)$ ,  $s \in [0, t]$  and initial probability density  $f(x, u, t = t_0)$

- ▶ average over all possible realizations of  $F_R(s)$ ,  $s \in [0, t]$  to get  $P(x, u, t)$ :

$$F(x, u, t) = \langle f(x, u, t) \rangle_{F_R}$$

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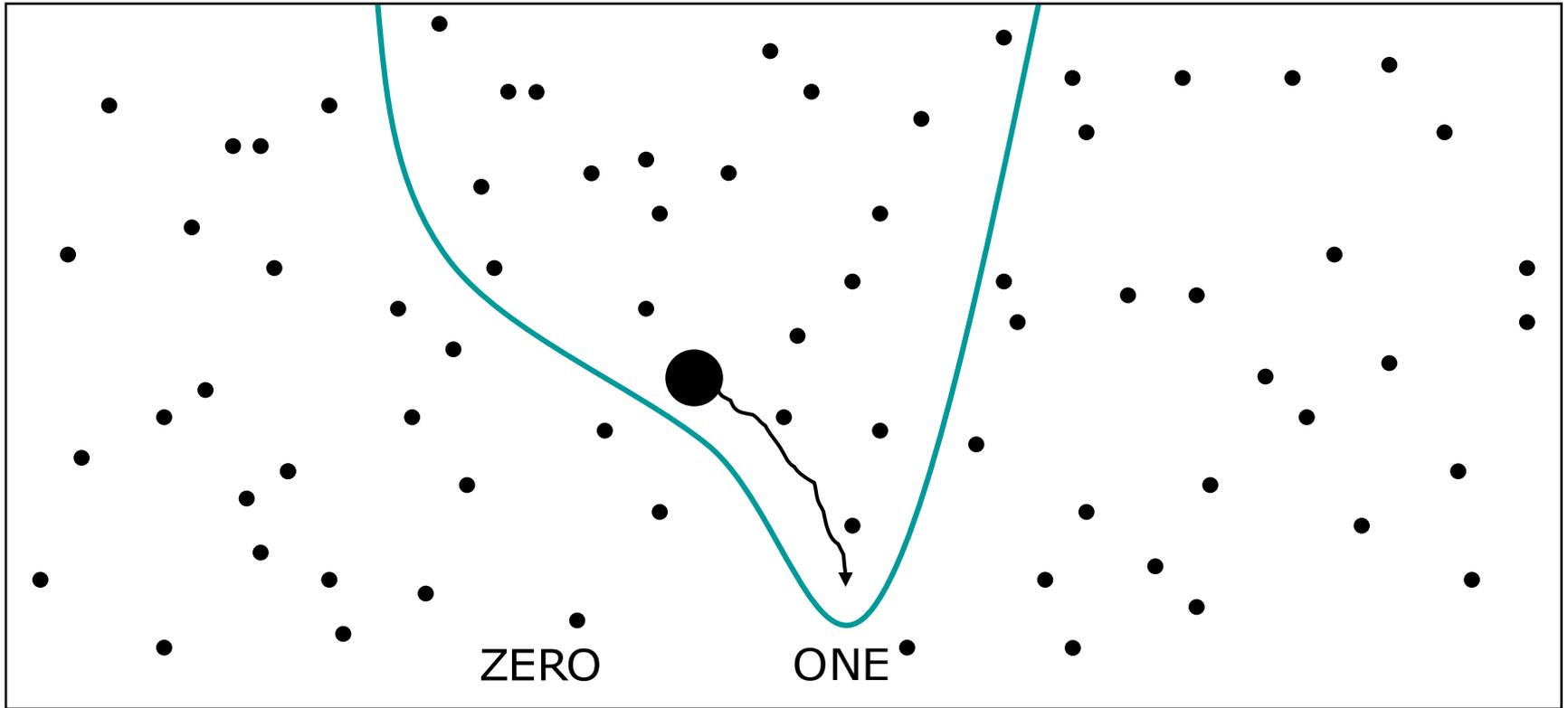
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Landauer's claim specified for stochastic model:

"No matter how one changes the potential with time to perform RTO, the average of the total heat generated during the operation will be not less than  $k_B T \ln 2$ "



$\langle X \rangle = \int f(x, u, t) X(x, u, t) dx du$       ensemble average of a function  $X$   
on the phase space

$\dot{Q}$  :      ensemble average of energy given to the particle by the  
environment per unit time

$\dot{W}$  :      ensemble average of work done by the potential  $V$   
on the particle per unit time

$S = - \int dx du f \ln f$  :      Shannon entropy

$$k_B = 1$$

choosespecificbistablepotential  $\rightarrow f_1(x, u)$  and  $f_0(x, u)$

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consider  $N$  ( $\gg 1$ ) memories:

$p_1 N$ : number of memories with value ONE

$p_0 N$ : number of memories with value ZERO

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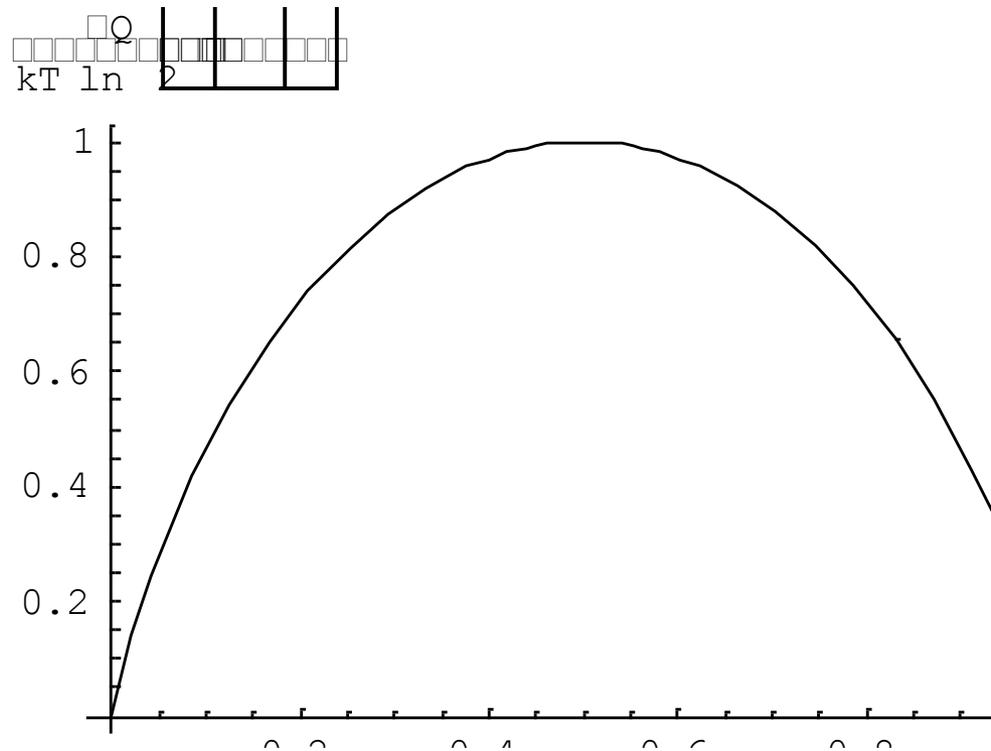
$p_0 N$ : number of memories with value ZERO

$\Delta Q_{out}(t_i, t_f)$ : average energy dissipated into the environment  
between times  $t_i$  and  $t_f$

$$\Delta Q_{out}(t_i, t_f) = \int_{t_i}^{t_f} (-\dot{Q}) dt \geq T [S(t_i) - S(t_f)]$$

# Discussion

- ▶ need noise force to be white and Gaussian
- ▶ consider  $\Delta Q(p_0, p_1)$  where  $p_1 = 1 - p_0$ :



## References

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