Exercise sheet X

due 13.5.2008.

Problem 1 [*Spinors*]: Let χ be a vector satisfying the eigenvector equation

$$\vec{e} \cdot \vec{\sigma}\, \chi = \chi \ ,$$

where $\vec{\sigma}$ are the Pauli matrices — see Problem 1 on Exercise sheet IX. We want to find the solution to this eigenvector equation by using the rotation symmetry of the problem.

Let χ_0 be the eivenvector

$$\left(\vec{e}_z \cdot \vec{\sigma}\right) \chi = \sigma_z \chi_0 = \chi_0 \ , \quad \text{i.e.} \qquad \chi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ .$$

Write

$$\vec{e} = R_z(\alpha) R_y(\beta)\, \vec{e}_z \ ,$$

where the two rotation matrices correspond to rotations around the y-axis and z-axis with angles β, α, respectively. Use your knowledge of problem 3 (iii), Exercise sheet VII to show that

$$\left[\left(R_y(\beta)^T R_z(\alpha)^T \vec{e}\right) \cdot \vec{\sigma}\right] \chi_0 = U_y^\dagger(\beta)\, U_z^\dagger(\alpha)\ \left(\vec{e} \cdot \vec{\sigma}\right)\ U_z(\alpha)\, U_y(\beta)\, \chi_0 \, .$$

Thus, in order to obtain χ, it is sufficient to just rotate the σ_z eigenspinor χ_0 in spinor space around the y and z-spinor axis by angles β, α, respectively. Calculate χ by using the expression obtained for $U(\omega \vec{n})$ in problem 3 (ii), Exercise sheet VII for general spinor rotations around axis $\vec{n}$ and angle ω.

Problem 2 [*Angular momentum*]: A particle is in the angular momentum eigenstate $\psi = |j, m\rangle = |1, 1\rangle$ with respect to the usual z-axis. Angular momentum with respect to the x-axis is measured.

(i) What are the possible outcomes of this measurement?

(ii) Calculate the expectation value of J_x in the above state ψ by writing J_x in terms of J^+ and J^-. Similarly, determine the expectation value of J_x^2 in this state. Deduce from these two results the probabilities for measuring $J_x = m$ for the different outcomes m.

Problem 3 [*Clebsch-Gordon coefficients*]: Derive the Clebsch-Gordon coefficients for the addition of spin-angular momenta of two spin 1/2 particles, $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ in an alternative way:

(i) Show that $\mathbf{S}^2$ can be written as

$$\mathbf{S}^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2 S_{1z} S_{2z} + S_{1+} S_{2-} + S_{1-} S_{2+} \, .$$

(ii) Find the 4×4 matrix A describing the action of $\mathbf{S}^2$ in the product basis $|\frac{1}{2}, m_1\rangle|\frac{1}{2}, m_2\rangle$.

(iii) Find the unitary matrix that diagonalises A. Explain why this transformation describes the change of basis from the product basis $|\frac{1}{2}, m_1\rangle|\frac{1}{2}, m_2\rangle$ to the basis of the total spin $|s, m\rangle$. Hence read off the corresponding Clebsch-Gordon coefficients.