Exercise sheet IX

due 6.5.2008.

Problem 1 [*Spinors*]: Let $\vec{e}$ be a fixed unit vector in $\mathbb{R}^3$. A spin 1/2 particle is in a state χ for which

$$\langle \vec{\sigma} \rangle = \vec{e}, \tag{1}$$

where $\vec{\sigma}$ are the Pauli matrices.

(i) First we want to determine the state χ. Show that eq. (1) implies that

$$(\vec{e} \cdot \vec{\sigma})\, \chi = \chi \ . \tag{2}$$

Then solve (2) as a straightforward eigenvalue problem for $\vec{e} \cdot \vec{\sigma}$, and calculate χ.

(ii) What is the probability that the measurement of the observable S_z (z-component of spin) in the above state yields the value $\hbar/2$.

Problem 2 [*Clebsch-Gordon coefficients*]:

(i) Prove that the Clebsch-Gordon coefficients for the decomposition of the tensor product of spin l and spin 1/2 to spin $l + 1/2$ are explicitly given by

$$\langle l, \tfrac{1}{2}, m \mp \tfrac{1}{2}, \pm\tfrac{1}{2} | l, \tfrac{1}{2}, l + \tfrac{1}{2}, m \rangle = \sqrt{\frac{l \pm m + \frac{1}{2}}{2l + 1}} \ .$$

Hint: consider expressions of the form $\langle j_1 j_2 m_1 m_2 | M_\pm | j_1 j_2 j m \rangle$ and let $M_\pm$ act to the left and to the right. Derive recursion relations and use induction on m. Start the induction by considering a state with $m = j - 1$. In order to identify states in the two bases look at states of highest spin.

(ii) An electron in the hydrogen atom is in an orbital angular momentum state with $l = 1$. Find the eigenstates of the total (orbital and spin) angular momentum $|j, j_z\rangle$ in $D_{\frac{3}{2}}$ in terms of those labelled by $|m, s\rangle$, where m and s are the z-components of the orbital and spin angular momentum operators, respectively.