Exercise sheet VI

due 15.4.2008.

Problem 1 [*Harmonic oscillator in external field*]: A particle of mass m moves along the x-axis under the influence of the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 + ex \ .$$

Show that the energy levels are

$$E_n = \left(N + \frac{1}{2}\right)\hbar\omega - \frac{1}{2}\frac{e^2}{m\omega^2} \ .$$

Problem 2 [*Harmonic oscillator in $d = 3$ dimensions*]: Consider a particle of unit mass now moving in $d = 3$ dimensions under the influence of the potential

$$V(\mathbf{x}) = \frac{1}{2}\sum_i^d \omega_i^2 x_i^2 \ .$$

(i) Solve the time-independent Schrödinger equation $H\psi(\mathbf{x}) = E\psi(\mathbf{x})$ for the corresponding Hamiltonian H, making the separation ansatz

$$\psi(\mathbf{x}) = \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \ .$$

(ii) Define the *generators*

$$E_{12} = -E_{21} = a_1^\dagger a_2 - a_2^\dagger a_1 \ , \quad E_{23} = -E_{32} = a_2^\dagger a_3 - a_3^\dagger a_2 \ , \quad E_{31} = -E_{13} = a_3^\dagger a_1 - a_1^\dagger a_3 \ .$$

Show that E_{12}, E_{23}, E_{31} satisfy the commutation relations of the Lie algebra $so(3)$,

$$[E_{12}, E_{23}] = E_{31} \ , \qquad [E_{23}], E_{31}] = E_{12} \ , \qquad [E_{31}, E_{12}] = E_{23} \ .$$

Hint: Determine first the commutation relations $[a_i, a_j^\dagger] = \delta_i^j$.

(iii) For the *isotropic case* $\omega_i \equiv \omega$, show that that $[H, E_{ij}] = 0$ ($i \neq j \in \{1,2,3\}$). Deduce that each E_{ij} maps an eigenstate of the Hamiltonoperator H with eigenvalue E_n into another eigenstate with the same eigenvalue.

Problem 3 [*Harmonic oscillator in $d = 2, 3$, degeneracies*]: Let D_n^k be the number of ways of writing n as the sum of k non-negative integers $n_1, n_2, \ldots, n_k$.

(i) Show that the generating function $F^k(s)$ can be written as

$$F^k(s) := \sum_{n=0}^\infty D_n^k s^n = \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \cdots \sum_{n_k=0}^\infty s^{n_1+\cdots+n_k} \ .$$

(ii) Deduce that $F^k(s) = (1-s)^{-k}$ and hence or otherwise show that

$$D_n^k = \binom{n+k-1}{n} \, .$$

(iii) Deduce the formulae for the degeneracy of energy eigenstates of the isotropic harmonic oscillators in two and three dimensions.

(iv) For $d = 3$, show that the energy eigenstates with the energy $E_{N=1} = \hbar\omega(1+3/2)$ form an irreducible, 3-dimensional representation of the Lie algebra so(3) (see Problem 2 (ii)). What is the situation for the states with $E_{N=2} = \hbar\omega(2 + 3/2)$?