Exercise sheet IV

due 1.4.2008.

Problem 1 [*Measurement, two-level-system*]: Consider a system given by the Hamiltonian

$$H = \begin{pmatrix} \varepsilon & -\gamma \\ -\gamma & \varepsilon \end{pmatrix} \tag{1}$$

where the two states $\psi_l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ may have the meaning of particle position in a box with two panels (*left* and *right*). The ψ_i $(i = l, r)$ can thus be considered as eigenstates of some panel operator A with eigenvalues ± 1.

(i) Compute the eigenvalues E_1, E_2 and eigenstates of H as well as the commutator $[H, A]$.

(ii) At $t = 0$ a position measurement yielded the state ψ_l (i.e. the particle was located in the left panel). Determine the probability that a energy measurement at $t = t'$ yields E_1 and E_2.

(iii) At $t = 0$ a position measurement yielded the state ψ_l. Determine the probability that at $t = t'$ a repeated position measurement yields ψ_i, $i = l, r$.

Problem 2 [*Square well potential, resonance, wave packet*]: Consider a square well potential $V(x) = -V_0$ for $|x| \le a$ and $V(x) = 0$ otherwise. We are considering an incident particle (with energy $E > 0$) from the left, scattering to a reflected and transmitted wave, i.e.

$$\begin{aligned} \psi(x) &= e^{ikx} + re^{-ikx} \quad \text{for } x < a \\ \psi(x) &= te^{ikx} \quad \text{for } x > a. \end{aligned}$$

(i) Defining $q = \sqrt{\hbar^2/2m(V_0 + E)}$, calculate the transmission amplitude t. [You may use your expression for t for the case of the potential barrier in problem 2, Exercise sheet III.]

(ii) The system is in resonance if $T = |t|^2 = 1$. Find the resonant energies E_R and momenta q_R, i.e. the values for which resonance takes place.

(iii) For $E - E_R \ll 1$ using a Taylor expansion, show that

$$t(E)e^{2ika} = (-1)^n \frac{i\Gamma/2}{E - E_R + i\Gamma/2}, \quad \text{with} \quad \frac{2}{\Gamma} = \left(\frac{1}{2}(\frac{q}{k} + \frac{k}{q}) \cdot \frac{\mathrm{d}}{\mathrm{d}E}(2qa) \right) |_{E_R} .$$

Rewrite $t(E)$ as $t(E) = |t(E)|e^{i\delta(E)}$.

(iv) Instead of one incident plane wave $\sim e^{ikx}$, we now consider an incoming wave *packet* $\psi_{\rm in}$, where

$$\psi_{\rm in}(x,t) = \int_0^\infty \mathrm{d}k f(k) \exp\left[i/\hbar(kx - E(k)t)\right].$$

For a general wave packet with phase function $(kx - E(k)t + \hbar\alpha(k))/\hbar$ and provided that $f(k)$ is narrowly peaked at $k = k_0$, the stationarity condition

$$\frac{d}{dk}(kx - E(k)t + \hbar\alpha(k))|_{k_0} = 0$$

defines the approximate center of the wave packet $x(t) = x_0 + v_0 t$, where $x_0 = -\hbar\frac{d}{dk}\alpha(k)|_{k_0}$ and the group velocity is $v_0 = \frac{d}{dk}E(k)|_{k_0}$. Calculate $x(t)$ given $\psi_{\rm in}$ above.

(v) Using the results from section (i) obtain the form of the transmitted (or scattered) wave *packet* $\psi_{\rm out}$. Write $t(E)$ as $t(E) = |t(E)|e^{i\delta(E)}$ (see (iii)), and derive an expression for $x(t)$ in this case.