Statistical Physics
Exercise Sheet 7

Exercise 7.1 Exact solution of the Ising chain

Consider the one-dimensional Ising model on a chain of length N with free ends

$$H = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}, \quad \sigma_i = \pm 1. \tag{1}$$

a) Compute the partition function Z_N using a recursive procedure.

b) Calculate the magnetization density $m = \langle \sigma_j \rangle$ where the spin σ_j is far away from the ends.

c) Show that the *spin correlation function* $\Gamma_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ decays exponentially on the scale of the *correlation length* $\xi = -[\log(\tanh \beta J)]^{-1}$ with increasing the distance $|j - i|$. What happens in the limit $T \to 0$?

d) Use the fluctuation relation in the thermodynamic limit $N \to \infty$,

$$\chi(T) = \frac{1}{k_B T} \left(\sum_j \langle \sigma_0 \sigma_j \rangle - N \langle \sigma_0 \rangle^2 \right), \tag{2}$$

to calculate the magnetic susceptibility in zero magnetic field.

Exercise 7.2 Exact solution of the Ising chain in the presence of a magnetic field

Consider the one-dimensional Ising model in the presence of a magnetic field h

$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i, \quad \sigma_i = \pm 1. \tag{3}$$

Assume periodic boundary conditions, that is, assume that the Nth spin is connected to the first so that the chain forms a ring. Then the spin labels run modulo N $(i + N = i)$. We will use the *transfer matrix* method to calculate the partition function Z_N in a nonzero magnetic field.

a) It is convenient to consider the bonds between two neighboring sites. Show that in this way the partition function can be written as

$$Z_N = \sum_{\{\sigma_i\}} P_{\sigma_1 \sigma_2} P_{\sigma_2 \sigma_3} \dots P_{\sigma_N \sigma_1} = \mathrm{Tr}\, \mathbf{P}^N.$$

The symmetric matrix $\mathbf{P}$ is called the transfer matrix. Find the entries of $\mathbf{P}$.

b) Show that in the thermodynamic limit $N \to \infty$ the free energy is given by

$$F(T, N, h) = -\frac{1}{\beta} \log Z_N = -N k_B T \log \left[e^{\beta J} \cosh \beta h + \sqrt{e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J}} \right] \tag{4}$$

c) Compute the magnetization $m(T, h) = \langle \sigma_0 \rangle$ and the susceptibility $\chi(T, h) = (\partial m / \partial h)_T$. Compare the results with the results of exercise 7.1.