Statistical Physics
Exercise Sheet 5

Exercise 5.1 Detailed balance of nuclear spin relaxation in a magnetic field

We consider the relaxation of nuclear spins in a magnetic field H_z due to the hyperfine interaction with conductance electrons. We assume independent nuclear spins $\vec{I}$ ($I = 1/2$). The interaction Hamiltonian is given by

$$\mathcal{H}_{int} = A\vec{I} \cdot \vec{S}\delta(\vec{r}) = A\delta(\vec{r}) \left[I_z S_z + \frac{1}{2}(I^+ S^- + I^- S^+) \right]$$

where $\vec{S}$ is the spin of the electron. In a magnetic field the energy of an electron $\epsilon_{\vec{k}\sigma}$ and the energy of a nuclear spin E_α are split by the Zeemann term

$$\epsilon_{\vec{k}\sigma} = \varepsilon_{\vec{k}} + 2\sigma\Delta_e, \quad \sigma = \pm\frac{1}{2}, \quad E_\alpha = 2\alpha\Delta_n, \quad \alpha = \pm\frac{1}{2}.$$

a) What are Δ_e and Δ_n? Compare the two quantities.

b) Assume free electrons and use Fermi's Golden rule to show that the total transition rate w_{+-} for processes $\alpha' = -1/2 \to \alpha = 1/2$ is given by

$$w_{+-} = \frac{A^2\pi}{2\hbar^3} \int d\varepsilon\, g\left(\varepsilon - \Delta_n + \Delta_e\right) g\left(\varepsilon + \Delta_n - \Delta_e\right) f(\varepsilon + \Delta_n) \left[1 - f(\varepsilon - \Delta_n)\right] \quad (1)$$

where $g(\varepsilon)$ is the electronic density of states per spin in the absence of a magnetic field and $f(\varepsilon)$ is the Fermi-Dirac distribution function. Find a similar expression for w_{-+}.

c) Show that

$$f(\varepsilon + \Delta_n) \left[1 - f(\varepsilon - \Delta_n)\right] = e^{-2\beta\Delta_n} f(\varepsilon - \Delta_n) \left[1 - f(\varepsilon + \Delta_n)\right].$$

Show that the principle of detailed balance in an applied magnetic field can therefore be stated as

$$\frac{w_{+-}(H_z)}{w_{-+}(H_z)} = e^{g_n \mu_n H_z / k_B T}. \quad (2)$$

Here, g_n is the gyromagnetic factor of the nuclei and μ_n the nuclear magneton.

d) Solve the master equations

$$\frac{dp_+}{dt} = w_{+-}p_- - w_{-+}p_+, \quad \frac{dp_-}{dt} = w_{-+}p_+ - w_{+-}p_-, \quad (3)$$

with initial conditions $p_+(0) = 1$ and $p_-(0) = 0$ for the probability p_α to find the nuclear spin in the state α. Use the principle of detailed balance as stated in c).

Exercise 5.2 AC conductivity of a metal

Calculate the conductivity for a time dependent electrical field $\vec{E}(t) = \vec{E}_0(\omega)e^{-i\omega t}$ using the constant relaxation time approximation to the Boltzmann equation. Assume for simplicity free electrons with dispersion $\varepsilon_{\vec{p}} = \vec{p}^2/2m$. Calculate the specific dissipation loss $P(\omega) = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} \mathrm{Re}[\vec{j}(t)] \cdot \mathrm{Re}[\vec{E}(t)] dt$