Statistical Physics
Exercise Sheet 3

Exercise 3.1 Local equilibrium state of a gas in a periodic potential

We consider a gas of N particles trapped in a box, $\vec{r} \in V = [0, L]^3$, in the presence of a conservative force $\vec{F}(\vec{r}) = -\nabla V(\vec{r})$ originating from a periodic potential in the x direction

$$V(\vec{r}) = V_0 \cos\left(\frac{2\pi x k}{L}\right), \quad k \in \mathbb{N}. \tag{1}$$

In the equilibrium the distribution function is given by

$$f_0(\vec{r}, \vec{p}) = \frac{n(\vec{r})}{(2\pi m k_B T)^{3/2}} e^{-\beta \frac{p^2}{2m}}, \quad \beta = \frac{1}{k_B T}. \tag{2}$$

a) Find the local density $n(\vec{r})$. Discuss the limits $V_0 \ll k_B T$ and $V_0 \gg k_B T$.

b) Determine the internal energy $U = \langle p^2/2m + V(\vec{r}) \rangle$ and the specific heat $C_V = (\partial U/\partial T)_V$. Discuss these expressions in the limits $V_0 \ll k_B T$ and $V_0 \gg k_B T$.

c) Calculate the entropy $S(T, V, N)$.

Hints: The integral representation and the series expansion of the modified Bessel functions of the first kind for integer order n are

$$I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos\theta} \cos(n\theta) d\theta = \left(\frac{z}{2}\right)^n \sum_{k \geq 0} \frac{(z^2/4)^k}{k!(n+k)!}, \quad n \in \mathbb{Z}. \tag{3}$$

The asymptotic behavior for $z \to \infty$ is

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left(1 - \frac{4n^2 - 1}{8z} + \dots\right).$$

Furthermore, the relation $I_0'(z) = I_1(z)$ holds.

Exercise 3.2 Maxwell-Boltzmann distribution function for relativistic particles

Find the equilibrium distribution function for relativistic particles of energy $E(\vec{p}) = \sqrt{p^2 c^2 + m^2 c^4}$, where m is the mass and c the speed of light. Show that in the limit $k_B T \ll mc^2$ the usual Maxwell-Boltzmann distribution function is recovered. Calculate the internal energy U and the specific heat C_V and find the first relativistic corrections to these expressions.

Hints: The integral representation of the modified Bessel functions of the second kind is

$$K_n(z) = \int_0^\infty e^{-z \cosh y} \cosh(ny) dy. \tag{4}$$

The asymptotic behavior for $z \to \infty$ is

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4n^2 - 1}{8z} + \frac{(4n^2 - 1)(4n^2 - 9)}{2!(8z)^2} + \frac{(4n^2 - 1)(4n^2 - 9)(4n^2 - 25)}{3!(8z)^3} + \dots\right]. \tag{5}$$