Solid State Theory
Solution 9

Problem 9.1 Distribution function

We consider a Fermi gas

$$H = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \frac{U}{\Omega} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\uparrow} c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\downarrow} c_{\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}\uparrow} \tag{1}$$

with a weak contact interaction $U \ll \epsilon_F$, such that we can treat the interaction term

$$V = \frac{U}{\Omega} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\uparrow} c^\dagger_{\boldsymbol{k}'-\boldsymbol{q}\downarrow} c_{\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}\uparrow} \tag{2}$$

as a perturbation to the free Fermi gas

$$H_0 = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}. \tag{3}$$

Hence, we expand the ground state of the interacting system

$$|\psi\rangle = |\psi^{(0)}\rangle + |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + \mathcal{O}(U^3) \tag{4}$$

around the non-interacting ground state

$$|\psi^{(0)}\rangle = |0\rangle = \prod_{\boldsymbol{k}<k_F,\sigma} c^\dagger_{\boldsymbol{k}\sigma} |\text{vac}\rangle \tag{5}$$

up to second order in the interaction.
Denoting the eigenstates and eigenenergies of the non-interacting system as $|m\rangle$ and E_m, respectively, such that $H_0|m\rangle = E_m|m\rangle$, and abbreviating the matrix elements of the interaction as $V_{ml} = \langle m|V|l\rangle$, the first- and second-order corrections to the ground state are in general

$$|\psi^{(1)}\rangle = \sum_{m\neq 0} \frac{V_{m0}}{E_0 - E_m} |m\rangle \tag{6}$$

$$|\psi^{(2)}\rangle = \sum_{m,l\neq 0} \frac{V_{ml}V_{l0}}{(E_0 - E_m)(E_0 - E_l)} |m\rangle - \sum_{m\neq 0} \frac{V_{00}V_{m0}}{(E_0 - E_m)^2} |m\rangle - \frac{1}{2} \sum_{m\neq 0} \frac{|V_{m0}|^2}{(E_0 - E_m)^2} |0\rangle. \tag{7}$$

Using these, the momentum distribution function is the expectation value of the occupation number operator $\hat{n}_{\boldsymbol{k}\sigma} = c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}$

$$\begin{aligned} n_{\boldsymbol{k}\sigma} = \langle\psi|\hat{n}_{\boldsymbol{k}\sigma}|\psi\rangle &= \langle\psi^{(0)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(0)}\rangle + \langle\psi^{(0)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(1)}\rangle + \langle\psi^{(1)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(0)}\rangle \\ &\quad + \langle\psi^{(1)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(1)}\rangle + \langle\psi^{(0)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(2)}\rangle + \langle\psi^{(2)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(0)}\rangle + \dots \qquad (8) \\ &= n^{(0)}_{\boldsymbol{k}\sigma} + \delta n^{(1)}_{\boldsymbol{k}\sigma} + \delta n^{(2)}_{\boldsymbol{k}\sigma} + \mathcal{O}(U^3). \qquad (9) \end{aligned}$$

The zeroth-order term is just the Fermi distribution at zero temperature, i.e. the step function

$$n^{(0)}_{\boldsymbol{k}\sigma} = n^0_k = \theta(\epsilon_F - \epsilon_{\boldsymbol{k}}). \tag{10}$$

The first-order term clearly vanishes

$$\delta n^{(1)}_{\boldsymbol{k}\sigma} = \langle\psi^{(0)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(1)}\rangle + \langle\psi^{(1)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(0)}\rangle = 0 \tag{11}$$

because the occupation number operator is diagonal in the non-interacting eigenbasis and the first-order correction (6) has no overlap with the non-interacting ground state, $\langle\psi^{(0)}|\psi^{(1)}\rangle = 0$. In second order, however, several terms contribute:

$$\begin{aligned}
\delta n^{(2)}_{\boldsymbol{k}\sigma} &= \langle\psi^{(1)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(1)}\rangle + \langle\psi^{(0)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(2)}\rangle + \langle\psi^{(2)}|\hat{n}_{\boldsymbol{k}\sigma}|\psi^{(0)}\rangle & (12)\\
&= \sum_{m,l\neq 0} \frac{V_{0m}V_{l0}}{(E_0-E_m)(E_0-E_l)}\langle m|\hat{n}_{\boldsymbol{k}\sigma}|l\rangle - \sum_{m\neq 0}\frac{|V_{m0}|^2}{(E_0-E_m)^2}\langle 0|\hat{n}_{\boldsymbol{k}\sigma}|0\rangle & (13)\\
&= \sum_{m\neq 0}\frac{|V_{m0}|^2}{(E_0-E_m)^2}\left(\langle m|\hat{n}_{\boldsymbol{k}\sigma}|m\rangle - \langle 0|\hat{n}_{\boldsymbol{k}\sigma}|0\rangle\right), & (14)
\end{aligned}$$

where in the last step we again used the fact that the occupation number operator is diagonal in the non-interacting basis.
The only states m for which the interaction has a non-vanishing matrix element V_{m0} differ by a pair of particle-hole excitations from the ground state. These can therefore be enumerated by summing over the momenta of the excitations

$$\sum_{m\neq 0}|m\rangle \to \sum_{\substack{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4\\ \boldsymbol{k}_1\neq\boldsymbol{k}_4}} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} c^\dagger_{\boldsymbol{k}_1\uparrow}c^\dagger_{\boldsymbol{k}_2\downarrow}c_{\boldsymbol{k}_3\downarrow}c_{\boldsymbol{k}_4\uparrow}|0\rangle = \sum_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4}{}' \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4} c^\dagger_{1\uparrow}c^\dagger_{2\downarrow}c_{3\downarrow}c_{4\uparrow}|0\rangle, \tag{15}$$

where we have used the abbreviations $c_{i\sigma} \equiv c_{\boldsymbol{k}_i,\sigma}$ and the primed sum indicates the omission of configurations with $\boldsymbol{k}_1 = \boldsymbol{k}_4$, which would be equal to the ground state $|0\rangle$. With this parameterization we are in a position to calculate the matrix elements

$$\begin{aligned}
V_{m0} &= \langle m|V|0\rangle = \langle 0|c^\dagger_{1\uparrow}c^\dagger_{2\downarrow}c_{3\downarrow}c_{4\uparrow}V|0\rangle & (16)\\
&= \frac{U}{\Omega}\sum_{\boldsymbol{k}_5,\ldots,\boldsymbol{k}_8}\delta_{\boldsymbol{k}_5+\boldsymbol{k}_6,\boldsymbol{k}_7+\boldsymbol{k}_8}\langle 0|c^\dagger_{1\uparrow}c^\dagger_{2\downarrow}c_{3\downarrow}c_{4\uparrow}c^\dagger_{5\uparrow}c^\dagger_{6\downarrow}c_{7\downarrow}c_{8\uparrow}|0\rangle & (17)\\
&= \frac{U}{\Omega}\langle 0|c^\dagger_{1\uparrow}c^\dagger_{2\downarrow}c_{3\downarrow}c_{4\uparrow}c^\dagger_{4\uparrow}c^\dagger_{3\downarrow}c_{2\downarrow}c_{1\uparrow}|0\rangle & (18)\\
&= \frac{U}{\Omega}\langle 0|\hat{n}_{1\uparrow}\hat{n}_{2\downarrow}(1-\hat{n}_{3\downarrow})(1-\hat{n}_{4\uparrow})|0\rangle & (19)\\
&= \frac{U}{\Omega}n^0_{k_1}n^0_{k_2}(1-n^0_{k_3})(1-n^0_{k_4}). & (20)
\end{aligned}$$

In the third line we used the fact that the matrix element vanishes unless all creation operators can be paired up with a corresponding destruction operator, then we have collected the pairs by using the canonic fermion commutation relations.
Next, the energy of the excited states in the non-interacting system is just given by the single-particle excitations with respect to the ground state, i.e.

$$E_m = E_0 + \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4, \tag{21}$$

where we again wrote $\epsilon_i \equiv \epsilon_{k_i}$.
Finally, using the commutation relations

$$[\hat{n}_i, c^\dagger_j] = \delta_{ij}c^\dagger_j, \qquad [\hat{n}_i, c_j] = -\delta_{ij}c_j, \tag{22}$$

which directly follow from the definition $\hat{n}_i = c_i^\dagger c_i$, the occupation number expectation value in the excited state is found to be

$$\begin{aligned} \langle m|\hat{n}_{\boldsymbol{k}\sigma}|m\rangle &= \langle m|\hat{n}_{\boldsymbol{k}\sigma} c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger c_{3\downarrow} c_{4\uparrow}|0\rangle & (23)\\ &= \langle m|c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger c_{3\downarrow} c_{4\uparrow}(\delta_{\boldsymbol{k},\boldsymbol{k}_1}\delta_{\sigma,\uparrow} + \delta_{\boldsymbol{k},\boldsymbol{k}_2}\delta_{\sigma,\downarrow} - \delta_{\boldsymbol{k},\boldsymbol{k}_3}\delta_{\sigma,\downarrow} - \delta_{\boldsymbol{k},\boldsymbol{k}_4}\delta_{\sigma,\uparrow} + \hat{n}_{\boldsymbol{k}\sigma})|0\rangle & (24)\\ &= \langle m|m\rangle(\delta_{\boldsymbol{k},\boldsymbol{k}_1}\delta_{\sigma,\uparrow} + \delta_{\boldsymbol{k},\boldsymbol{k}_2}\delta_{\sigma,\downarrow} - \delta_{\boldsymbol{k},\boldsymbol{k}_3}\delta_{\sigma,\downarrow} - \delta_{\boldsymbol{k},\boldsymbol{k}_4}\delta_{\sigma,\uparrow} + \hat{n}^0_{\boldsymbol{k}\sigma}) & (25)\\ &= \delta_{\sigma,\uparrow}(\delta_{\boldsymbol{k},\boldsymbol{k}_1} - \delta_{\boldsymbol{k},\boldsymbol{k}_4}) + \delta_{\sigma,\downarrow}(\delta_{\boldsymbol{k},\boldsymbol{k}_2} - \delta_{\boldsymbol{k},\boldsymbol{k}_3}) + \hat{n}^0_{\boldsymbol{k}\sigma}. & (26)\end{aligned}$$

Inserting all the parts into Eq. (14), we have

$$\begin{aligned} \delta n^{(2)}_{\boldsymbol{k}\sigma} &= \frac{U^2}{\Omega^2}\sum_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4}{}' \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}\frac{|n^0_{k_1}n^0_{k_2}(1-n^0_{k_3})(1-n^0_{k_4})|^2}{(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)^2}\\ &\quad\times\left(\delta_{\sigma,\uparrow}(\delta_{\boldsymbol{k},\boldsymbol{k}_1} - \delta_{\boldsymbol{k},\boldsymbol{k}_4}) + \delta_{\sigma,\downarrow}(\delta_{\boldsymbol{k},\boldsymbol{k}_2} - \delta_{\boldsymbol{k},\boldsymbol{k}_3}) + \hat{n}^0_{\boldsymbol{k}\sigma} - \hat{n}^0_{\boldsymbol{k}\sigma}\right), & (27)\\ &= \frac{U^2}{\Omega^2}\sum_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4}{}' \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{k}_3+\boldsymbol{k}_4}\frac{n^0_{k_1}n^0_{k_2}(1-n^0_{k_3})(1-n^0_{k_4})}{(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)^2}\\ &\quad\times\left(\delta_{\sigma,\uparrow}(\delta_{\boldsymbol{k},\boldsymbol{k}_1} - \delta_{\boldsymbol{k},\boldsymbol{k}_4}) + \delta_{\sigma,\downarrow}(\delta_{\boldsymbol{k},\boldsymbol{k}_2} - \delta_{\boldsymbol{k},\boldsymbol{k}_3})\right). & (28)\end{aligned}$$

Inspection of the last expression shows that, depending on the external momentum and spin arguments $\boldsymbol{k}$ and σ, only one pair of Kronecker δ's in the large parenthesis will make a finite contribution: Consider, e.g., $\sigma = \uparrow$ and $k < k_F$; then the large parenthesis reduces to $(\delta_{\boldsymbol{k},\boldsymbol{k}_1} - \delta_{\boldsymbol{k},\boldsymbol{k}_4})$, but for $\boldsymbol{k} = \boldsymbol{k}_4$ the factor $(1 - n^0_{k_4}) = 0$, so that only the term with $\boldsymbol{k} = \boldsymbol{k}_1$ remains. We hence find

$$\delta n^{(2)}_{\boldsymbol{k}\sigma} = \frac{U^2}{\Omega^2}\sum_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_3}\frac{\delta_{\boldsymbol{k}+\boldsymbol{k}_1,\boldsymbol{k}_2+\boldsymbol{k}_3}}{(\epsilon_k+\epsilon_1-\epsilon_2-\epsilon_3)^2}\times\begin{cases} n^0_{k_1}(1-n^0_{k_2})(1-n^0_{k_3}), & k<k_F\\ -(1-n^0_{k_1})n^0_{k_2}n^0_{k_3}, & k>k_F.\end{cases} \quad (29)$$

Eliminating the summation over $\boldsymbol{k}_3$ with the Kronecker δ and replacing the momentum sums by integrals as usual, we have for the first case, $k < k_F$,

$$\begin{aligned} \delta n^{(2)-}_{\boldsymbol{k}\sigma} &= U^2\int_{k_1<k_F}\frac{d^3k_1}{(2\pi)^3}\int_{k_2>k_F}\frac{d^3k_2}{(2\pi)^3}\frac{(1-n^0_{k_3})}{(\epsilon_k+\epsilon_1-\epsilon_2-\epsilon_3)^2} & (30)\\ &= \frac{4m^2U^2}{(2\pi)^6\hbar^4}\int_{k_1<k_F}d^3k_1\int_{k_2>k_F}d^3k_2\frac{\theta(k_3-k_F)}{[k^2+k_1^2-k_2^2-k_3^2]^2} & (31)\end{aligned}$$

and correspondingly for the second case, $k > k_F$,

$$\delta n^{(2)+}_{\boldsymbol{k}\sigma} = \frac{4m^2U^2}{(2\pi)^6\hbar^4}\int_{k_1>k_F}d^3k_1\int_{k_2<k_F}d^3k_2\frac{\theta(k_F-k_3)}{[k^2+k_1^2-k_2^2-k_3^2]^2}. \quad (32)$$

In order to evaluate the integrals we now introduce new variables

$$\boldsymbol{p} = \frac{\boldsymbol{k}+\boldsymbol{k}_1}{k_F} = \frac{\boldsymbol{k}_2+\boldsymbol{k}_3}{k_F}, \qquad \boldsymbol{q} = \frac{\boldsymbol{k}_2-\boldsymbol{k}_3}{k_F}. \quad (33)$$

For simplicity we also fix the external momentum to the Fermi surface, so we can write $\boldsymbol{k} = k_F\hat{\boldsymbol{k}}$. With these definitions the boundary conditions for the integrals translate to

$$p = \frac{|\boldsymbol{k}+\boldsymbol{k}_1|}{k_F} < \frac{|k_F+k_F|}{k_F} = 2, \qquad |\boldsymbol{p}-\hat{\boldsymbol{k}}| = \frac{\boldsymbol{k}_1}{k_F} < 1, \quad (34)$$

$$|\boldsymbol{p}+\boldsymbol{q}| = \frac{2\boldsymbol{k}_2}{k_F} > 2, \qquad |\boldsymbol{p}-\boldsymbol{q}| = \frac{2\boldsymbol{k}_3}{k_F} > 2. \quad (35)$$

We thus need to compute

$$\delta n^{(2)}_{k_F^-} = \frac{2m^2U^2k_F^2}{(2\pi)^6\hbar^4}\int_{p<2}\mathrm{d}^3p\int\mathrm{d}^3q\frac{\theta(1-|\boldsymbol{p}-\hat{\boldsymbol{k}}|)\theta(|\boldsymbol{p}\pm\boldsymbol{q}|-2)}{\left[(\boldsymbol{p}-2\hat{\boldsymbol{k}})^2-q^2\right]^2}, \tag{36}$$

where a factor of $k_F^6/8$ is the Jacobian from the change of variables and a factor of $4/k_F^4$ was extracted from the integrand's denominator. As the whole system is isotropic, the value of the full expression cannot depend on the direction of the momentum $\boldsymbol{k}$. We can therefore average $\delta n^{(2)}_{k_F^-}$ over all directions by integrating over $\hat{\boldsymbol{k}}$ and dividing by the solid angle 4π. Changing the order of integrations, we then perform this integral before the others, i.e. we integrate the integrand for fixed $\boldsymbol{p},\boldsymbol{q}$ and choose ϑ to be the angle between the vectors $\boldsymbol{p}$ and $\hat{\boldsymbol{k}}$:

$$\frac{1}{4\pi}\int\mathrm{d}^2\hat{\boldsymbol{k}}\frac{\theta(1-|\boldsymbol{p}-\hat{\boldsymbol{k}}|)}{\left[(\boldsymbol{p}-2\hat{\boldsymbol{k}})^2-q^2\right]^2} = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\vartheta\sin\vartheta\frac{\theta(1-(p^2+1-2p\cos\vartheta)^{1/2})}{\left[p^2+4-q^2-4p\cos\vartheta\right]^2} \tag{37}$$

$$= \frac{1}{2}\int_{-1}^{1}\mathrm{d}t\frac{\theta(1-(p^2+1-2pt))}{\left[p^2+4-q^2-4pt\right]^2}, \tag{38}$$

where in the last step we substituted $t=\cos\vartheta$ and used $\theta(1-\sqrt{x})=\theta(1-x)$ for $x>0$. The step function vanishes for $t<p/2$, therefore we can account for it by adjusting the integration range accordingly and obtain

$$\frac{1}{2}\int_{-1}^{1}\mathrm{d}t\frac{\theta(1-(p^2+1-2pt))}{\left[p^2+4-q^2-4pt\right]^2} = \frac{1}{2}\int_{p/2}^{1}\mathrm{d}t\frac{1}{\left[p^2+4-q^2-4pt\right]^2} \tag{39}$$

$$= \frac{1}{4}\frac{p-2}{[(p-2)^2-q^2](p^2+q^2-4)}. \tag{40}$$

Now

$$\delta n^{(2)}_{k_F^-} = \frac{(mUk_F)^2}{2(2\pi)^6\hbar^4}\int_{p<2}\mathrm{d}^3p\int\mathrm{d}^3q\,\theta(|\boldsymbol{p}\pm\boldsymbol{q}|-2)\frac{p-2}{[(p-2)^2-q^2](p^2+q^2-4)} \tag{41}$$

only the step functions depend on the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$. We therefore change to spherical coordinates $\int\mathrm{d}^3q\to\int_0^\infty\mathrm{d}q\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi\mathrm{d}\vartheta\,q^2\sin\vartheta$ and choose the z-axis to point in the direction of $\boldsymbol{p}$ such that ϑ denotes the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$. Then the integral over ϕ is trivial and we need to compute

$$\int_0^\pi\mathrm{d}\vartheta\sin\vartheta\,\theta(|\boldsymbol{p}+\boldsymbol{q}|-2)\theta(|\boldsymbol{p}-\boldsymbol{q}|-2)$$

$$= \int_0^\pi\mathrm{d}\vartheta\sin\vartheta\,\theta(\sqrt{p^2+q^2+2pq\cos\vartheta}-2)\theta(\sqrt{p^2+q^2-2pq\cos\vartheta}-2) \tag{42}$$

$$= \int_0^\pi\mathrm{d}\vartheta\sin\vartheta\,\theta(p^2+q^2+2pq\cos\vartheta-4)\theta(p^2+q^2-2pq\cos\vartheta-4) \tag{43}$$

$$= \int_{-1}^{1}\mathrm{d}t\,\theta(p^2+q^2-4+2pqt)\theta(p^2+q^2-4-2pqt) \tag{44}$$

$$= \int_{-1}^{1}\mathrm{d}t\,\theta(\frac{p^2+q^2-4}{2pq}+t)\theta(\frac{p^2+q^2-4}{2pq}-t) \tag{45}$$

$$= \int_{-1}^{1}\mathrm{d}t\,\theta(x+t)\theta(x-t),\qquad x=\frac{p^2+q^2-4}{2pq}. \tag{46}$$

Since the first step function equals one for $t > -x$ and the second one for $t < x$, their product vanishes for all t if $x < 0$. If, on the other hand, $x > 1$ then the integrand is equal to one over the whole integration range $-1 < t < 1$. For $0 < x < 1$, finally, $\int_{-1}^{1} \mathrm{d}t\, \theta(x+t)\theta(x-t) = \int_{-x}^{x} \mathrm{d}t 1 = 2x$. We therefore find

$$\int_0^\pi \mathrm{d}\vartheta \sin\vartheta\, \theta(|\boldsymbol{p}+\boldsymbol{q}|-2)\theta(|\boldsymbol{p}-\boldsymbol{q}|-2) \quad = \begin{cases} 0, & q < \sqrt{4-p^2}, \\ \frac{p^2+q^2-4}{pq}, & \sqrt{4-p^2} < q < p+2, \\ 2, & q > p+2. \end{cases} \tag{47}$$

We hence arrive at

$$\begin{aligned} \delta n_{k_F^-}^{(2)} &= \frac{(mUk_F)^2}{2(2\pi)^5\hbar^4} \int_{p<2} \mathrm{d}^3p \Big[\int_{\sqrt{4-p^2}}^{p+2} \mathrm{d}q\, \frac{q^2(p-2)}{[(p-2)^2-q^2]pq} \\ &\qquad + \int_{p+2}^{\infty} \mathrm{d}q\, \frac{2q^2(p-2)}{[(p-2)^2-q^2](p^2+q^2-4)} \Big] \qquad (48) \\ &= \frac{(mUk_F)^2}{(2\pi\hbar)^4} \int_0^2 \mathrm{d}p \Big[\int_{\sqrt{4-p^2}}^{p+2} \mathrm{d}q\, \frac{pq(p-2)}{(p-2)^2-q^2} \\ &\qquad + \int_{p+2}^{\infty} \mathrm{d}q\, \frac{2p^2q^2(p-2)}{[(p-2)^2-q^2](p^2+q^2-4)} \Big] \qquad (49) \\ &= \frac{(mUk_F)^2}{(2\pi\hbar)^4} \int_0^2 \mathrm{d}p \Big[\frac{p}{2}(p-2)\ln\frac{2-p}{4} \\ &\qquad + p\sqrt{4-p^2}\tanh^{-1}\frac{p+2}{\sqrt{4-p^2}} + \frac{p}{2}(p-2)\ln\frac{2}{p} + i\pi\frac{p}{2}\sqrt{4-p^2} \Big] \qquad (50) \\ &= \frac{(mUk_F)^2}{(2\pi\hbar)^4} \left[\frac{1}{9}(5+6\ln 2) + \frac{1}{9}(1+12\ln 2) \right] \qquad (51) \\ &= \frac{(mUk_F)^2}{(2\pi\hbar)^4} \left(\frac{2}{3} + 2\ln 2 \right). \qquad (52) \end{aligned}$$

The computation of $\delta n_{k_F^+}^{(2)}$ is similar to that of $\delta n_{k_F^-}$. The main difference is that we invert the sign of the arguments of the Heaviside θ-functions. Thus, using the definitions in Eq. (33), the boundary conditions given in Eq. (35) become

$$p = \frac{|\boldsymbol{k}+\boldsymbol{k}_1|}{k_F} > \frac{|k_F+k_F|}{k_F} = 2, \qquad |\boldsymbol{p}-\hat{\boldsymbol{k}}| = \frac{\boldsymbol{k}_1}{k_F} > 1, \tag{53}$$

$$|\boldsymbol{p}+\boldsymbol{q}| = \frac{2\boldsymbol{k}_2}{k_F} < 2, \qquad |\boldsymbol{p}-\boldsymbol{q}| = \frac{2\boldsymbol{k}_3}{k_F} < 2, \tag{54}$$

and we have to compute

$$\delta n_{k_F^+}^{(2)} = -\frac{2m^2U^2k_F^2}{(2\pi)^6\hbar^4} \int_{p>2} \mathrm{d}^3p \int \mathrm{d}^3q \frac{\theta(|\boldsymbol{p}-\hat{\boldsymbol{k}}|-1)\theta(2-|\boldsymbol{p}\pm\boldsymbol{q}|)}{\left[(\boldsymbol{p}-2\hat{\boldsymbol{k}})^2-q^2\right]^2}. \tag{55}$$

Carrying out the integration over $\hat{\mathbf{k}}$ as before, we obtain

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}t \frac{\theta((p^2+1-2pt)-1)}{[p^2+4-q^2-4pt]^2} = \frac{1}{4}\frac{2+p}{[(p+2)^2-q^2](4-p^2-q^2)}, \tag{56}$$

and the corresponding result of the integration in Eq. (46) is

$$\int_{-1}^{1} \mathrm{d}t\, \theta(-x+t)\theta(-x-t), \qquad x = \frac{p^2+q^2-4}{2pq}. \tag{57}$$

Through similar argumentation that leads to Eq. (58)

$$\int_{-1}^{1} \mathrm{d}\vartheta \sin\vartheta\, \theta(2-|\boldsymbol{p}+\boldsymbol{q}|)\theta(2-|\boldsymbol{p}-\boldsymbol{q}|) \quad = \begin{cases} 0, & q > \sqrt{4-p^2}, \\ \frac{4-p^2-q^2}{pq}, & \sqrt{4-p^2} > q > 2-p, \\ 2, & q < 2-p. \end{cases} \tag{58}$$

Therefore,

$$\delta n^{(2)}_{k_F^+} = -\frac{(mUk_F)^2}{(2\pi\hbar)^4} \int_0^2 \mathrm{d}p \Big[\int_{\sqrt{4-p^2}}^{p-2} \mathrm{d}q\, \frac{pq(p+2)}{(p+2)^2-q^2}$$

$$+ \int_0^{p-2} \mathrm{d}q\, \frac{2p^2q^2(p+2)}{[(p+2)^2-q^2](4-p^2-q^2)} \Big] \tag{59}$$

$$= -\frac{(mUk_F)^2}{(2\pi\hbar)^4} \left(-\frac{2}{3} + 2\ln 2 \right). \tag{60}$$

As a result, the size of the discontinuity of the density distribution at the Fermi energy is reduced to

$$\delta n_{k_F^-} - \delta n_{k_F^+} = 1 - \frac{(mUk_F)^2}{4(\pi\hbar)^4} \ln 2. \tag{61}$$

In three dimensions, we can use the density of states at the Fermi energy

$$N(\epsilon_F) = \frac{3}{2}\frac{n}{\epsilon_F} = \frac{mk_F}{(\pi\hbar)^2} \tag{62}$$

(since $n = k_F^3/(3\pi^2)$) to simplify the result to its final form

$$\delta n_{k_F^-} - \delta n_{k_F^+} = 1 - \left(\frac{UN(\epsilon_F)}{2} \right)^2 \ln 2. \tag{63}$$