Chapter 1

Quantum Teleportation

- Quantum teleportation is one of the most important protocols in QIP it is simple yet powerful and elegant. It also spectacularly shows that quantum entanglement could be a resource enabling tasks that are otherwise not possible.
- Quantum teleporation (QT) allows us to send a qubit, that is transmit its full information content, without physically sending it. Most importantly, the task is achieved without either sender or receiver actually measuring the qubit; In other words, they remain ignorant to the information content during the transmission of the qubit.
- Note that carrying out such a procedure for classical bits is straightforward. The difficulty in the case of a qubit stems from:
  - Measurements on a single quantum system does not allow us to faithfully reconstruct/determine an initially unknown wave-function.
  - It is impossible to make a perfect copy of an unknown quantum state.

Theorem: (Quantum no-cloning – Wootters & Zurek 1982)
There is no unitary transformation that satisfies

$$|\Psi_A\rangle |0_B\rangle \xrightarrow{U} |\Psi_A\rangle |\Psi_B\rangle \quad \forall\Psi \in \varepsilon_A$$

Proof: Assume that there is unitary transformation U that copies arbitrary quantum states:

$$\begin{aligned} |\Psi\rangle |0_B\rangle |0_C\rangle &\xrightarrow{U} |\Psi_A\rangle |\Psi_B\rangle |e_C\rangle \\ |\varphi\rangle |0_B\rangle |0_C\rangle &\rightarrow |\varphi_A\rangle |\varphi_B\rangle |f_C\rangle \end{aligned}$$

Here qubit B is the target qubit and C is an auxiliary set of qubits.

→ Since U must preserve inner product, we require:

$$\langle\Psi|\varphi\rangle_A = \langle\Psi|\varphi\rangle_A \langle\Psi|\varphi\rangle_B \langle e|f\rangle_C \tag{1.1}$$

→ For orthonormal states ($\langle\Psi|\varphi\rangle = 0$), 1.1 is trivially satisfied.

→ If $1 > \langle\Psi|\varphi\rangle > 0$, we require

$$\underbrace{\langle\Psi|\varphi\rangle_B \langle e|f\rangle_C = 1!}_{\text{impossible since state vectors are normalized.}} \qquad \square$$

⇒ We cannot rely on copying for teleportation; in fact the protocol we find should make sure that the original qubit information is destroyed, so that the whole procedure is consistent with the no-cloning theorem.

QT protocol: Alice communicates the quantum information carried by her qubit to Bob.

1. Alice and Bob should share an entangled pair of qubits – say in state $|\Phi^+\rangle_{AB}$. If they have quantum memory, this entanglement sharing could be done well in advance.

2. Alice uses the qubit A of $|\Phi^+\rangle_{AB}$ and the qubit to be teleported $|\varphi\rangle_C$, to carry out a Bell measurement. This measurement projects the qubits A & B onto a maximally entangled state i.e. $|\Phi^\pm\rangle_{CA}$, $|\varphi^\pm\rangle_{CA}$.

   - qubit C no longer carries the quantum information.

3. Alice sends the result of her Bell measurement (that is the 2 classical bits – parity & phase) to Bob via a classical communication channel.

4. Bob applies a unitary transformation on qubit B; the transformation is determined by the parity & phase information from Alice:

$$\begin{array}{lcl} \left|\Phi^+\right\rangle_{CA} & \to & \mathbb{1}_B \\ \left|\Phi^-\right\rangle_{CA} & \to & \sigma_z^B \\ \left|\Psi^+\right\rangle_{CA} & \to & \sigma_x^B \\ \left|\Psi^-\right\rangle_{CA} & \to & \sigma_y^B \end{array}$$

→ Ideally, the protocol is deterministic and yields a fidelity $F = 1$.

→ Let's assume $|\varphi\rangle_C = \alpha|0\rangle + \beta|1\rangle$; the initial state is then:

$$\begin{aligned} |\Phi\rangle_{CAB} &= (\alpha|0\rangle_C + \beta|1\rangle_C)\,\frac{1}{\sqrt{2}}\,(|00\rangle_{AB} + |11\rangle_{AB}) \\ &= \frac{1}{\sqrt{2}}\,(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) \end{aligned}$$

Substituting $|00\rangle = \frac{1}{\sqrt{2}}\,(|\Phi^+\rangle + |\Phi^-\rangle)$, etc., we get:

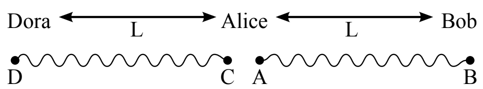
$$
\begin{aligned}
|\Phi\rangle_{CAB} = \frac{1}{2}\Big[&\alpha\left(|\Phi^+\rangle_{CA} + |\Phi^-\rangle_{CA}\right)|0\rangle_B \quad + \quad \beta\left(|\Psi^+\rangle_{CA} - |\Psi^-\rangle_{CA}\right)|0\rangle_B \\
&+ \alpha\left(|\Psi^+\rangle_{CA} + |\Psi^-\rangle_{CA}\right)|1\rangle_B \quad + \quad \beta\left(|\Phi^+\rangle_{CA} - |\Phi^-\rangle_{CA}\right)|1\rangle_B\Big] \\
= \frac{1}{2}\Big[& |\Phi^+\rangle_{CA} \otimes (\alpha|0\rangle_B + \beta|1\rangle_B) \quad + \quad |\Phi^-\rangle_{CA} \otimes (\alpha|0\rangle_B - \beta|1\rangle_B) \\
&+ |\Psi^+\rangle_{CA} \otimes (\beta|0\rangle_B + \alpha|1\rangle_B) \quad + \quad |\Psi^-\rangle_{CA} \otimes (-\beta|0\rangle_B + \alpha|1\rangle_B)\Big] \\
= \frac{1}{2}\Big[& |\Phi^+\rangle|\varphi\rangle_B + |\Phi^-\rangle\hat{\sigma}_z|\varphi\rangle_B + |\Psi^+\rangle\sigma_x|\varphi\rangle_B + |\Psi^-\rangle(-i\hat{\sigma}_y)|\varphi\rangle_B\Big]
\end{aligned}
$$

- Since $\sigma_i^2 = \mathbb{1} \quad \forall i$, projection of C & A onto a maximally entangled state, followed by the local unitary on qubit B (step 4) ensures successful QT.

- In contrast to a direct quantum communication protocol where a qubit is physically sent, QT is superior since quantum information is not subjected to decoherence in transmission (assuming Alice and Bob verify that their entangled qubit pairs are in a maximally entangled state).

1.1 Entanglement swapping

So far, we have assumed that the qubit to be teleported is in a pure state. This need not be the case; in fact the qubit C that Alice wants to teleport can be maximally entangled with another qubit D, held by Dora.



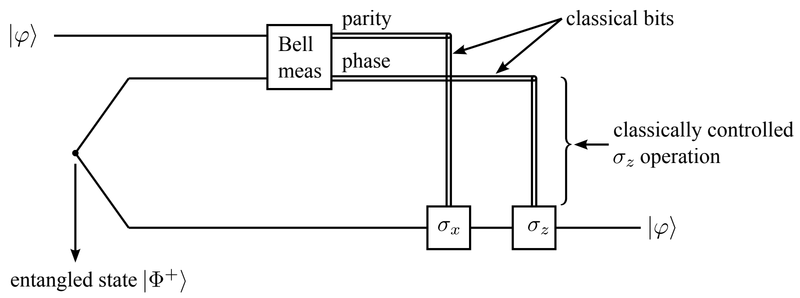
Applying the QT protocol yields

$$
\begin{aligned}
|\Phi^\pm\rangle_{DC}|\Phi^+\rangle_{AB} &\xrightarrow{\text{QT}} |\Phi^\pm\rangle_{DB} \\
|\Psi^\pm\rangle_{DC}|\Phi^+\rangle_{AB} &\xrightarrow{\text{QT}} |\Psi^\pm\rangle_{DB}
\end{aligned}
$$

→ the procedure is called entanglement swapping:
starting out with 2 maximally entangled states separated each by L, we end up with a single entangled pair separated by $2L$.

→ Key element in quantum repeaters.

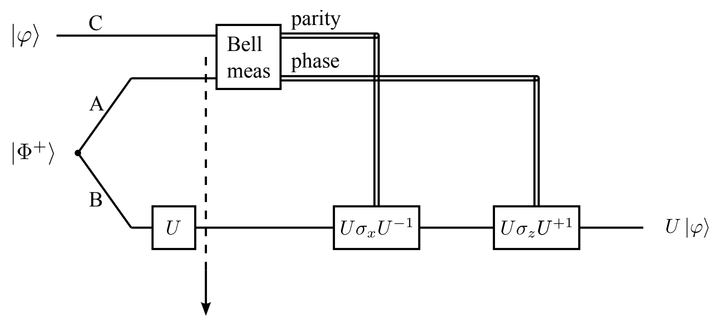
1.2 Circuit description of QT:



1.3 Quantum gate teleportation

QT protocol can be modified such that the teleported state is not the input state $|\varphi\rangle$, but rather $U|\varphi\rangle$.

Consider the circuit:



can be considered as teleportation using

$U_B\left|\Phi^+\right\rangle$ $\Leftrightarrow$ also a maximally entangled state since local operation cannot change the amount of entanglement.

$\rightarrow$ Useful if implementing U fault tolerantly is hard, but $U\sigma_x U^{-1}$ is easy.

1.4 Quantum dense coding

A protocol that allows us to send 2 classical bits by sending one qubit – once again, this is possible if Alice and Bob share a maximally entangled qubit pair.

1. Alice and Bob have previously shared $|\Phi^+\rangle$ state – in principle long before Alice acquired the information she wants to transmit.

2. Alice manipulates her qubit before sending

$$
\begin{array}{lllll}
00 & \rightarrow & \text{Alice applies} & \mathbb{1}_A & ; \quad \mathbb{1}\left|\Phi^+\right\rangle = \left|\Phi^+\right\rangle \\
01 & \rightarrow & \text{Alice applies} & \sigma_x^A & ; \quad \sigma_x^A\left|\Phi^+\right\rangle = \left|\Psi^+\right\rangle \\
10 & \rightarrow & \text{Alice applies} & \sigma_y^A & ; \quad \sigma_y^A\left|\Phi^+\right\rangle = -i\left|\Psi^-\right\rangle \\
11 & \rightarrow & \text{Alice applies} & \sigma_z^A & ; \quad \sigma_z^A\left|\Phi^+\right\rangle = \left|\Psi^-\right\rangle
\end{array}
$$

3. Upon receiving qubit A from Alice, Bob carries out a Bell measurement on A & B. The parity & phase bits he obtains are the 2 classical bits Alice transmitted.

- Note that the 2-bit message Alice sent is highly confidential. If Eve intercepts, she extracts no information as $\hat{\rho}_A = \frac{1}{2}\mathbb{1}$.