Chapter 1

Quantum Entanglement

- Enganglement is a feature of quantum mechanics that plays a central role in many quantum information processing (QIP) protocols:
  - → Two quantum systems that are in an entangled state exhibit strong correlations with no classical analog.
  - → While research aimed at generation, classification and detection of quantum entanglement blossomed with the advent of QIP, it is now recognized that entanglement is of central interest in condensed matter many body physics.

Definition: Consider a composite (bipartite) quantum system, composed of systems A & B: the composite system is entangled if its state-vector cannot be written in the form

$$\underbrace{|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle}_{\text{separable state}} \tag{1.1}$$

- Since it may not always be easy to see if $|\Psi_{AB}\rangle$ can be written in form (1.1), it is useful to consider the *Schmidt decomposition*:

Theorem: Let $|\Psi_{AB}\rangle$ be any pure state of a composite bipartite system. Then there exists orthonormal basis $\{|i_A\rangle\}$ of system A and $\{|i_B\rangle\}$ for system B, such that

$$|\Psi_{AB}\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \tag{1.2}$$

The number of non-zero λ_i's in the expansion (1.2) is called the Schmidt number $= R$;

If $R > 1$, the bipartite state is entangled.

→ However, $R = 2$ both for $|\phi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ and for $\left|\tilde{\phi}\right\rangle = |\uparrow\downarrow\rangle - \epsilon|\downarrow\uparrow\rangle$. In that sense, the Schmidt number does not quantify entanglement except from distinguishing product states from the rest.

1.1 How does a bipartite system become entangled?

1. Interactions: evolution under a Hamiltonian of the form:

$$H = \sum_i \hat{A}_i \otimes \hat{B}_i \Rightarrow \text{deterministic}$$

ex 1: Two confined spins with Heisenberg exchange.

$$\underbrace{H_1 = J_1 \vec{\sigma}^A \cdot \vec{\sigma}^B}_{|\Psi_s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\uparrow\downarrow\rangle) \text{ is the ground state of } H_s}$$

ex 2: Two confined spins, initially in $|\uparrow\downarrow\rangle$ subject to transverse exchange:

$$H_2 = J_2 \left(\sigma_x^A \sigma_x^B + \sigma_y^A \sigma_y^B\right)$$
$$\Rightarrow \underbrace{|\Psi(t)\rangle = \alpha(t)\,|\uparrow\downarrow\rangle + \beta(t)\,|\downarrow\uparrow\rangle}_{\text{evolution under } H_2}$$

with $\alpha(t) \neq 0 \neq \beta(t)$ for most t.

2. Projection based on a certain measurement outcome:
⇒ Probabilistic, does not require interactions.

ex 3:

$$|\Psi\rangle = (|\uparrow_A\rangle + \epsilon\,|\downarrow_A\rangle\,|1_a\rangle) \otimes (|\uparrow_B\rangle + \epsilon\,|\downarrow_B\rangle\,|1_b\rangle)$$

where $|1_{a,b}\rangle$ denotes a photon in mode a,b. If we detect a photon in mode $\hat{c} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{b}\right)$, then post-measurement state is:

$$|\Psi_{\text{post}}\rangle \cong \frac{1}{\sqrt{2}}\left(|\uparrow\rangle\,|\downarrow\rangle + |\downarrow\rangle\,|\uparrow\rangle\right)$$

Here we rely on erasure of the which path information.

1.2 Where does entanglement show up?

→ Ground-states of condensed-matter systems:

- Fractional quantum Hall state (Topological order)
- Metals with magnetic impurities (Kondo effect)

→ Decoherence of quantum systems:
when a quantum system A is entangled with system R and we have no access to R, we end up with decoherence of A.

→ Quantum error correcting codes:
deliberate/controlled generation of entanglement between quantum systems/bits to fight off entanglement of the unwanted sort (decoherence).

→ Photons generated in parametric down conversion (useful – teleporation).

→ In QIP we typically use time-dependent Hamiltonians where we control the time-evolution of a target quantum system A using another quantum system (such as a laser) which we treat classically; quantum effects in the control will result in (unwanted) entanglement.

1.3 Measures of entanglement

- von Neumann entropy (pure states)
- Bell's inequality violation
- Concurrence
- Entanglement witnesses

Preliminaries:

So far we have considered pure bipartite states where we have complete information about the composite system. In practice, this is typically not the case and we have to describe the system using density operators:

- The most general description of a quantum system is:

$$\underbrace{\hat{\rho}_A = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|}_{\text{with probability } p_i \text{ the system is in } \left|\phi_i\right\rangle}$$

  - state is pure when $p_i \neq 0$ only for a single i.
  - for mixed states, the decomposition is not unique.
  - A mixed-state of a composite system is separable if it can be written as:
  $$\hat{\rho}_{AB} = \sum_j p_j \hat{\rho}_A^j \otimes \hat{\rho}_B^j \tag{1.3}$$
  where $\hat{\rho}_A^j$, $\hat{\rho}_B^j$ are proper density matrices. (otherwise it is entangled.)
  - Given $\hat{\rho}_{AB}$, the reduced system density operator is:
  $$\hat{\rho}_A \triangleq \mathrm{Tr}_B \left\{\hat{\rho}_{AB}\right\}$$

- Classical vs quantum correlations:

  → The state $\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$ is an uncorrelated state.

  → The separable state (1.3) exhibits classical correlations: whenever system A has $\hat{\rho}_A^j$, system B has $\hat{\rho}_B^j$.

  → The state $\left|\Psi\right\rangle = \frac{1}{\sqrt{2}} \left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)$ exhibits quantum correlations. Whatever direction we choose to measure the 2 spins, we always find that they are pointing in opposite directions.

In contrast $\rho_{AB} = \frac{1}{2} \left|\uparrow\downarrow\right\rangle \left\langle\uparrow\downarrow\right| + \frac{1}{2} \left|\downarrow\uparrow\right\rangle \left\langle\downarrow\uparrow\right|$ exhibits correlations in a single direction.

→ More generally, given 2 distant quantum systems, classical correlations are those that can be generated using local (quantum) operators and classical communication (LOCC).

Ex: Alice and Bob have access (only) to quantum systems A & B. They share a random number generator; whenever the number is i, they prepare the state $\hat{\rho}_A^i \otimes \hat{\rho}_B^i$. If the probability of outcome i is p_i, the density operator describing the ensemble they generate is:

$$\hat{\rho}_{AB} = \sum_i p_i \hat{\rho}_A^i \otimes \hat{\rho}_B^i.$$

- For a pure entangled state, we have complete information about the system yet our information about individual systems is incomplete. Information is hidden in the quantum correlations. To see this point more clearly, we introduce maximally entangled states.

Definition A bipartite qubit state is maximally entangled if

$$\hat{\rho_A} \triangleq \mathrm{Tr}_B\left(\hat{\rho}_{AB}\right) = \frac{1}{2}\mathbb{1} = \mathrm{Tr}_A(\hat{\rho}_{AB}) = \hat{\rho}_B$$

that is the reduced density operator of either of the qubits contain no information, i.e. whichever basis we use to measure the qubit, the two outcomes are equally likely ($p_1 = P_2 = 1/2$)

→ For the bipartite qubit system, there are 4 orthonormal maximally entangled states: ($\left|\uparrow\right\rangle \leftrightarrow \left|0\right\rangle$; $\left|\downarrow\right\rangle \leftrightarrow \left|1\right\rangle$)

$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle \pm \left|11\right\rangle\right) \tag{1.4a}$$

$$\left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle \pm \left|10\right\rangle\right) \tag{1.4b}$$

These maximally entangled states can be classified according to their

1. Parity: i.e. are the two qubits/spins aligned ($\left|\Phi^{\pm}\right\rangle$) or anti-aligned ($\left|\Psi^{\pm}\right\rangle$).
2. Relative phase: i.e. is the state in-phase ($\left|\Phi^{+}\right\rangle, \left|\Psi^{+}\right\rangle$) or out-of-phase ($\left|\Phi^{-}\right\rangle, \left|\Psi^{-}\right\rangle$).

⇒ Specifying the parity and relative-phase completely determines the maximally entangled state; both of these two bits of information refer to correlations between the spins and tells us nothing about individual spin orientations.

⇒ Just as we can carry out measurements that project the 2-qubit state onto $\{\left|00\right\rangle, \left|01\right\rangle, \left|10\right\rangle, \left|11\right\rangle\}$ we could carry out joint measurements projecting onto $\{\left|\Phi^{\pm}\right\rangle, \left|\Psi^{\pm}\right\rangle\}$. First, quick reminder:

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $|0\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow |\uparrow\rangle$

$\hat{\rho}_A = \frac{1}{2}\left(\mathbb{1} + \vec{P} \cdot \vec{\sigma}\right)$ with $\left|\vec{P}\right| \leq 1$ (Bloch sphere).

- Consider the operators $\sigma_x^A \otimes \sigma_x^B$ & $\sigma_z^A \otimes \sigma_z^B$.

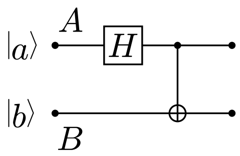
  1. They can be measured simultaneously:
  $$\left[\sigma_x^A \sigma_x^B, \sigma_z^A \sigma_z^B\right] = 0$$
  2. $\sigma_z^A \sigma_z^B$ measures parity:
  $$\begin{aligned} \sigma_z^A \sigma_z^B \left|\Phi^\pm\right\rangle &= \left|\Phi^\pm\right\rangle \\ \sigma_z^A \sigma_z^B \left|\Psi^\pm\right\rangle &= -\left|\Psi^\pm\right\rangle \end{aligned}$$
  3. $\sigma_x^A \sigma_x^B$ measures relative phase:
  $$\begin{aligned} \sigma_x^A \sigma_x^B \begin{matrix} \left|\Phi^+\right\rangle \\ \left|\Psi^+\right\rangle \end{matrix} &= \begin{matrix} \left|\Phi^+\right\rangle \\ \left|\Psi^+\right\rangle \end{matrix} \\ \sigma_x^A \sigma_x^B \begin{matrix} \left|\Phi^-\right\rangle \\ \left|\Psi^-\right\rangle \end{matrix} &= -\begin{matrix} \left|\Phi^-\right\rangle \\ \left|\Psi^-\right\rangle \end{matrix} \end{aligned}$$

  $\rightarrow$ Just as the simultaneous measurement of σ_z^A, σ_z^B projects the state vector onto $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, a simultaneous measurement of $\sigma_z^A \sigma_z^B$, $\sigma_x^A \sigma_x^B$ projects onto $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\}$ and yields the parity and relative phase information.

- How do we carry out such a measurement?

  Recall the Hadamard $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ gates:

  the circuit:

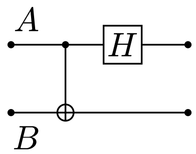


realizes the map:

$$\begin{array}{lll} \overset{ab}{|00\rangle} \to & \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)|0\rangle \to & \left|\Phi^+\right\rangle \\ |01\rangle \to & \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)|1\rangle \to & \left|\Psi^+\right\rangle \\ |10\rangle \to & \quad\quad\quad\quad\quad\quad\quad \to & \left|\Phi^-\right\rangle \\ |11\rangle \to & \quad\quad\quad\quad\quad\quad\quad \to & \left|\Psi^-\right\rangle \end{array}$$

Since this circuit describes a unitary evolution, we can run it backwards to achieve the map:

$$\begin{array}{llll} \left|\Phi^+\right\rangle \to & |00\rangle\,; & \left|\Psi^+\right\rangle \to & |01\rangle\,; \\ \left|\Phi^-\right\rangle \to & |10\rangle\,; & \left|\Psi^-\right\rangle \to & |11\rangle\,; \end{array}$$



+ measurement of σ_z^A & σ_z^B at the output is equivalent to measurement of $\sigma_x^A \sigma_x^B$ & $\sigma_z^A \sigma_z^B$.

- Note however that this measurement requires interactions (i.e. CNOT gate) and cannot be carried out deterministically if the qubits are sparated.

- An obvious measure for the degree of entanglement of a composite pure state is given by the von Neumann entropy of the reduced density operator:

$$S_A = \mathrm{Tr}\left\{\hat{\rho}_A \log_2 \hat{\rho}_A\right\}$$

  where

$$\log_2 \hat{\rho}_A = 1.44 \left[\left(\hat{\rho}_A - \hat{\mathbb{1}}\right) - \frac{\left(\hat{\rho}_A - \hat{\mathbb{1}}\right)^2}{2} + \cdots \right]$$

  $\to$ If $\hat{\rho}_A$ is diagonal in an orthonormal basis:

$$\underbrace{S_A = \sum_i p_i \log_2 p_i}$$

$S_A = 0 \leftrightarrow$ pure state

$S_A = 1 \leftrightarrow$ maximally entangled state.

→ We say that a maximally entangled 2-qubit state has 1 ebit[1] of entanglement.

Bell's inequalities:

A seemingly very different and experimentally accessible way to determine if a composite quantum system is entangled, is to carry out a Bell inequality experiment to determine if it is violated.

→ This measure applies to pure and mixed states.

→ Significance of Bell's inequality violation goes way beyond determining the presence of entanglement, since it shows that nature (obeying quantum mechanics) is incompatible with the predictions of local realism (classical physics).

Realism: physical properties/observables have definite values independent of observation.

Locality: If two physical systems are space-like separated, then an action/measurement performed on A cannot modify the physical description (i.e. measurement results/probabilities) of system B.

- Einstein, Podolsky and Rosen (EPR) envisioned that quantum mechanics is incomplete and that there must be a more general theory, involving parameters not currently experimentally accessible (hidden variables), that satisfies local realism.
- Bell has shown that even the most general classical local hidden variable theory (LHVT) should satisfy inequalities that are violated by quantum mechanics.

CHSH inequality (a variant on the original Bell inequality).
Premise:

- Charlie prepares 2 (twin) particles and sends one to Alice and the other to Bob. This process is repeated many times.
- Alice measures two 2 physical properties, Q and R. These physical observables produce/give an outcome that is either $+1$ or -1.

$$Q = \pm 1 \qquad\qquad R = \pm 1$$

  The values of Q & R for each particle are assumed to be objective properties, determinedly fixed by the values of hidden variables. Since Alice has no access to hidden variables, her results are probabilistic.
- Bob measures physical properties S and T

$$S = \pm 1 \qquad\qquad T = \pm 1$$

[1]An ebit is a two-party quantum state with quantum entanglement and the fundamental unit of bipartite entanglement

- Alice & Bob are space-like separated and they decide on which measurement to make immediately prior to the measurement. The decision and measurement is done on a timescale short compared to L_{AB}/c.
- Since Q, R and S, T are objective properties of the twin particles, the quantity
$$Q \cdot S + R \cdot S + R \cdot T - Q \cdot T$$
has a definite value for a given set of hidden variables.
$$QS + RS + RT - QT = (Q + R) \cdot S + (R - Q) \cdot T$$
Since $R, Q = \pm 1$, either $Q + R = 0$ or $R - Q = 0$. In addition since $S, T = \pm 1$, we necessarily have
$$QS + RS + RT - QT = \pm 2. \tag{1.5}$$
- Now, we assume that $p(q, r, s, t)$ is the probability that the system of twin-particles is found in a state where $Q = q$, $R = r$, $S = s$, $T = t$. The probabilities are ultimately determined by the hidden variables that Charlie (who prepares the twin particles) does not have access to.

$\Rightarrow$ A sequence of experiments on many twin-particles can be used to determine the expectation value of the observable $QS + RS + RT - QT$:

$$\begin{aligned} \mathbb{E}\left(QS + RS + RT - QT\right) &= \sum_{q,r,s,t} p(q, r, s, t)(qs + rs + rt - qt) \\ &\overset{\text{using (1.5)}}{\leq} 2 \sum_{q,r,s,t} p(q, r, s, t) = 2, \end{aligned} \tag{1.6}$$

In addition,

$$\begin{aligned} \mathbb{E}\left(QS + RS + RT - QT\right) &= \sum_{q,r,s,t} p(q, r, s, t)qs + \sum_{q,r,s,t} p(q, r, s, t)rs \\ &+ \sum_{q,r,s,t} p(q, r, s, t)rt - \sum_{q,r,s,t} p(q, r, s, t)qt \\ &= \mathbb{E}(QS) + \mathbb{E}(RS) + \mathbb{E}(RT) - \mathbb{E}(QT) \leq 2, \end{aligned} \tag{1.7}$$

where, in the first equation, we use the fact that p is independent of which observable is measured.

$\therefore$ If a LHVT description could be used to describe all physical systems, then the CHSH inequality (1.7) would never be violated.

- We can now consider an analogous setting in quantum mechanics:
  - Charlie prepares $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(\overset{AB}{|00\rangle} + \overset{AB}{|11\rangle} \right)$ and sends qubit A (B) to Alice (Bob).

- By tossing a coin each time, Alice decides to measure:

$$T = \sigma_z^A$$
or
$$R = \sigma_x^A$$

- Bob will also make a completely random choice to measure

$$S = \left(\sigma_z^B + \sigma_x^B\right)/\sqrt{2}$$
or
$$T = \left(-\sigma_z^B + \sigma_x^B\right)/\sqrt{2}$$

-

$$\begin{aligned}\left\langle \Phi^+ \left| QS \right| \Phi^+ \right\rangle &= \frac{1}{\sqrt{2}} \langle \Phi^+ | \underbrace{\sigma_z^A \sigma_z^B}_{1} + \underbrace{\sigma_z^A \sigma_x^B}_{\text{yields } 0} | \Phi^+ \rangle \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Similarly $\langle RS \rangle = \frac{1}{\sqrt{2}} = \langle RT \rangle$ and $\langle QT \rangle = -\frac{1}{\sqrt{2}}$

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2} > 2.$$

$\Rightarrow$ $|\Phi^+\rangle$ violates Bell-CHSH inequality. Note that $\{|\Phi^-\rangle, |\Psi^+\rangle\}$ do not violate this particular inequality. However by changing the measured observables, we can find a Bell inequality violated by each entangled pure state.

- Note that the expectation values of type

$$\left\langle \phi \left| \sigma_z^A \otimes \sigma_z^B \right| \phi \right\rangle$$

can be obtained either by a joint 2-qubit measurement of $\sigma_z^A \otimes \sigma_z^B$ (as we described earlier - possible only with interactions) or by independent measurements of σ_z^A & σ_z^B. The latter destroys the relative phase information since $\left[\sigma_z^A, \sigma_x^A \sigma_x^B\right] \neq 0$.

- Experiments in a variety of physical systems (photons, ions, spins) have been shown to violate Bell's inequalities, suggesting that either localism or realism (or both) have to be abandoned.

Entanglement witnesses: For every entangled state $\hat{\rho}_e$, there is an observable W such that

$$\langle W \rangle_{\hat{\rho}_e} = \mathrm{Tr}\left\{\rho_e W\right\} < 0$$

and $\mathrm{Tr}\left\{\rho_{\mathrm{sep}} W\right\} > 0$ for all separable states $\hat{\rho}_{\mathrm{sep}}$.

$\Rightarrow$ By measuring W and finding a negative expectation value, we can determine that the system is entangled.

$\Rightarrow$ Bell's inequalities could be cast in the form of an entanglement witness.